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Wealth Creation, Wealth Dilution and Population Dynamics

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JEL classification codes
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Abstract

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1 Introduction

Is demography destiny? How economic development interacts with population dynamics is a key question that has gained renewed interest in the profession: demographic change lies at the origin of many challenges faced by modern societies, and the widespread decline of fertility rates in the industrialized world affects policymaking on a global scale. In the economics literature, there is an established body of theories that endogeneize private fertility choices and their response to economic growth (Bloom et al. 2013). Yet, the majority of these models – with exceptions that we discuss below – predict that the economy converges to long-run equilibria where population grows exponentially at a constant rate, which is at odds with the demographers view of the long run (Wrigley, 1988). In this paper, we propose a model where the interaction between individual fertility choices and productivity growth led by innovations delivers a theory of the population level: in the long run, output grows at a positive endogenous rate, while the mass of population achieves a constant endogenous level. The nature of this steady state, and the determinants of long-run consumption and income shares are in stark contrast with the conclusions of the existing literature.

The two building blocks of our model are the overlapping-generations demographic structure and the Schumpeterian theory of R&D-based endogenous growth. In the demographic block of the model, we assume that households optimize lifetime consumption facing a positive probability of death (Yaari, 1965; Blanchard, 1985), and we extend this framework to endogeneize fertility: households choose the number of children by maximizing own utility subject to a pure time cost of reproduction. The demographic structure implies that per capita consumption growth is limited by *financial wealth dilution*: since disconnected generations optimize over finite horizons, each new cohort entering the economy captures a fraction of the existing wealth by pursuing independent accumulation decisions that reduce the consumption possibilities of all generations.[1] The novel insight of our model is that wealth dilution interacts with fertility choices and is thus both a consequence and a determinant of population dynamics. Specifically, financial wealth dilution tends to reduce the economy's fertility rate by limiting consumption expenditures. The key question is whether this negative partial-equilibrium effect translates into a general-equilibrium negative feed-

[1] Buiter (1988) and Weil (1989) provide an early recognition of the wealth dilution effect in the Blanchard-Yaari framework with exogenous population growth.

back of population on fertility. The answer hinges on how the supply side generates growth in financial assets, and more precisely, on how population affects the economy's rate of *wealth creation*.

We model the production side of the economy as an R&D-based model of endogenous growth where assets represent ownership of firms. The key characteristic of the process of wealth creation is that both the mass of firms and the average profitability of each firm grow endogenously as a result of different R&D activities (Peretto, 1998; Peretto and Connolly, 2007). Growth in the total value of firms results from both vertical innovations (i.e., each individual firm invests in R&D that raises internal productivity) and horizontal innovations (i.e., new firms enter the market) that compete for labor as an input. In equilibrium, the ratio between the market wage rate and assets per capita is increasing in population size. This relationship propagates the effect of wealth dilution, giving rise to a negative feedback of population on fertility that eventually brings population growth to a halt in the long run. Hence, unlike standard growth models predicting exponential population growth,[2] we obtain a theory of the population level where net fertility is asymptotically zero despite a positive endogenous rate of output growth.

Our results differ from those of the existing literature in two major ways. First, with respect to alternative models admitting a constant endogenous population level in the long run, our distinctive hypothesis is that production possibilities are *not* constrained by resource scarcity. To the best of our knowledge, all the existing theories of the population level hinge on Malthusian mechanisms whereby population is bounded by the scarcity of essential factors available in fixed supply. Malthusian mechanisms may take various forms: decreasing returns to scale in Eckstein et al. (1988), land scarcity combined with subsistence requirements in Galor and Weil (2000), open-access livestock in Brander and Taylor (1998). Two borderline cases are Strulik and Weisdorf (2008) and Peretto and Valente (2015), where the fixed factor is a marketed input and its relative scarcity creates price effects that tend to reduce fertility through increased cost of living and/or reduced real incomes.[3] Our present analysis, instead, fully abstracts from Malthusian mechansism by

[2] The class of growth models with endogenous fertility predicting exponential population growth is quite large (see Ehrlich and Lui, 1997) and encompasses all the well-established specifications of the supply side, from neoclassical technologies (e.g., Barro and Becker, 1989) to endogenous growth frameworks (e.g., Chu et al. 2013).

[3] In Strulik and Weisdorf (2008), scarcity increases the relative price of food and thereby the private

ruling out fixed endowments: the predicted fertility decline originates in the dilution of financial wealth. Moreover, we abstract from the research questions typically tackled by Malthusian models – prominently, the rise of and escape from pre-industrial stagnation traps – to address inherently forward-looking issues.

The second main difference with respect to the existing literature is the determination of key macroeconomic variables in the long run. In the steady state of our model, the ratios of aggregate consumption and of total labor incomes to aggregate financial wealth are exclusively determined by demography and preference parameters. This is in stark contrast with traditional balanced-growth models – especially those assuming a neoclassical supply-side structure – where the same 'long-run ratios' are fundamentally determined by technology. A major implication of our result is that exogenous shocks hitting demographic or preference parameters – and by extension, public policies affecting reproduction costs, life expectancy, or migration – have a first-order impact on income shares, growth and welfare. We perform a quantitative analysis of the model that delivers further insights in this respect. We calibrate the model parameters using data for OECD countries, and we assess the impact of exogenous demographic shocks on consumption, growth, and welfare using numerical simulations. Negative demographic shocks caused by higher reproduction costs and immigration barriers induce permanent reductions in labor income shares while having a 'reversed impact' on growth rates: growth may be higher in the very long run, but it is lower during the transition. This phenomenon originates in the positive co-movement between population and mass of firms, and bears substantial consequences for welfare. Using a cohort-specific index of lifetime utility, we show that a permanent shock reducing net immigration by 25% reduces welfare for all the generations entering the economy up to eighty years after the shock, due to the combined effects of permanently lower labor incomes and stagnating transitional growth.

The paper is organized as follows. After describing the demographic model (section 2) and the production side of the economy (section 3), we characterize the steady state with constant population in section 4. Section 5 derives key analytical results and extends the model to include migration. Section 6 presents our quantitative analysis, and Section 7

cost of reproduction resulting in fertility decline. In Peretto and Valente (2015), growing rents from scarce natural resources affect fertility via income effects that, assuming substitutability between land and labor inputs in production, give rise to a stationary population level that is nonetheless determined by resource scarcity via resource prices.

concludes. To preserve expositional clarity, we report detailed derivations and proofs in a separate Appendix.

2 The Demographic Model

The economy is populated by overlapping generations of single-individual families facing a constant probability of death (Yaari, 1965; Blanchard, 1985). We extend the Blanchard-Yaari structure by assuming that fertility is endogenously determined by private benefits and costs: each household derives utility from the mass of children it rears subject to a pure time cost of reproduction. Since wealth is not transmitted in a dynastic fashion, the arrival of new individuals dilutes financial wealth per capita, affecting consumption possibilities as well as population dynamics via fertility choices.

2.1 Households

The economy's population consists of different cohorts of single-individual families indexed by their birth date j. Individual variables take the form $x_j(t)$, where $j \in (-\infty, t)$ is the cohort index and $t \in (-\infty, \infty)$ is continuous calendar time.[4] In particular, $c_j(t)$ denotes consumption at time t of an individual born at time $j < t$, and $b_j(t)$ denotes the mass of children reared at time t by an agent who belongs to cohort j. The expected lifetime utility of an individual born at time j is

$$U_j^E = \int_j^{\infty} \left[\ln c_j(t) + \psi \ln b_j(t)\right] e^{-(\rho+\delta)(t-j)} dt, \tag{1}$$

where $\psi > 0$ is the weight attached to the utility from rearing $b_j(t)$ children, $\rho > 0$ is the rate of time preference, and $\delta > 0$ is the constant instantaneous probability of death. Differently from dynastic models with pure altruism (e.g., Becker and Barro, 1988), individuals do not maximize their descendants' utility via intergenerational transfers. Children leave the family immediately after birth, enter the economy as workers owning zero assets, and make plans independently from their predecessors. Individuals accumulate assets and allocate one unit of time between working and child-rearing activities. The individual budget constraint is

$$\dot{a}_j(t) = (r(t) + \delta) a_j(t) + (1 - \gamma b_j(t)) w(t) - p(t) c_j(t), \tag{2}$$

[4] Using standard notation, the time-derivative of variable $x_j(t)$ is $\dot{x}_j(t) \equiv \mathrm{d}x_j(t)/\mathrm{d}t$.

where a_j is individual asset holdings, r is the rate of return on assets, w is the wage rate, p is the price of the consumption good, and $\gamma > 0$ is the time cost of child rearing per child. The term $(1 - \gamma b)$ thus represents individual labor supply.

An individual born at time j maximizes (1) subject to (2), taking the paths of all prices as given. Necessary conditions for utility maximization are the individual Euler equation for consumption

$$\frac{\dot{c}_j(t)}{c_j(t)} + \frac{\dot{p}(t)}{p(t)} = r(t) - \rho, \tag{3}$$

and the condition equating the marginal rate of substitution between consumption and child-rearing to the ratio of the respective marginal costs,

$$\frac{1/c_j(t)}{\psi/b_j(t)} = \frac{p(t)}{\gamma w(t)}, \qquad \text{or} \qquad b_j(t) = \frac{\psi}{\gamma} \cdot \frac{p(t)\, c_j(t)}{w(t)}, \tag{4}$$

where γw is the private opportunity cost of reproduction in terms of foregone labor income. The second expression in (4) emphasizes that individual fertility is proportional to the ratio between consumption expenditure and the market wage rate in each instant, a result that will play an important role in our analysis.

2.2 Aggregation and Population Dynamics

Denoting by $k_j(t)$ the size of cohort j at time t, adult population $L(t)$ and total births $B(t)$ equal

$$L(t) \equiv \int_{-\infty}^{t} k_j(t)\, dj \quad \text{and} \quad B(t) \equiv \int_{-\infty}^{t} k_j(t)\, b_j(t)\, dj.$$

Similarly, total assets $A(t)$ and aggregate consumption $C(t)$ equal

$$A(t) \equiv \int_{-\infty}^{t} k_j(t)\, a_j(t)\, dj \quad \text{and} \quad C(t) \equiv \int_{-\infty}^{t} k_j(t)\, c_j(t)\, dj.$$

Following the tradition of the literature, we define *per capita* variables by referring to adult population $L(t)$ as to the economy's population: births, assets, and consumption per capita are respectively denoted by $b \equiv B/L$, $a \equiv A/L$, and $c \equiv C/L$.[5] Since individuals are homogeneous within cohorts, the size of each cohort declines over time at rate δ, which

[5] The caveat in order is that population is in fact $L + B$, so that variables per capita should in principle be defined as the aggregate variables divided by $L + B$. Doing so would complicate the algebra without changing the results.

therefore represents the economy's mortality rate. Population evolves according to the demographic law

$$\dot{L}(t) = B(t) - \delta L(t). \tag{5}$$

Total births are determined by households' reproduction choices: aggregating the individual fertility decision (4) across cohorts, we have

$$B(t) = \frac{\psi}{\gamma} \cdot \frac{p(t) \int_{-\infty}^{t} k_j(t) c_j(t) dj}{w(t)} = \frac{\psi}{\gamma} \cdot \frac{p(t) C(t)}{w(t)}. \tag{6}$$

Result (6) is a static equilibrium relationship between the economy's gross fertility, consumption expenditure and wages. Fertility and consumption dynamics will be subject to the wealth constraint: aggregation of the individual budget (2) across cohorts yields the growth rate of total assets

$$\frac{\dot{A}(t)}{A(t)} = r(t) + \frac{w(t)(L(t) - \gamma B(t))}{A(t)} - \frac{p(t) C(t)}{A(t)}, \tag{7}$$

where the term $L - \gamma B$ equals aggregate net labor supply and captures the negative impact of reproduction costs on the pace of accumulation.

2.3 Consumption and Wealth Dilution

We characterize individual consumption by exploiting the standard definition of human wealth,

$$h(t) \equiv \int_{t}^{\infty} w(s) \cdot e^{-\int_t^s (r(v)+\delta) dv} ds. \tag{8}$$

Combining the fertility equation (4) with the budget constraint (2), we obtain the individual expenditure of an agent born at time j,

$$p(t) c_j(t) = \frac{\rho + \delta}{1 + \psi} \cdot [a_j(t) + h(t)]. \tag{9}$$

Expression (9) shows that individual expenditure is proportional to individual wealth, given by the sum of current financial and human wealth. Financial wealth a_j is cohort-specific whereas human wealth h only depends on the anticipated paths of the wage and the interest rates. The existence of a preference for children, ψ, reduces the individual propensity to consume out of wealth. Integrating individual expenditures across cohorts and dividing by the population level, we can write per capita consumption expenditure as

$$p(t) c(t) = \frac{\rho + \delta}{1 + \psi} \cdot [a(t) + h(t)]. \tag{10}$$

Despite their apparent similarity, expressions (9) and (10) represent different objects. In the individual expenditure function, both $c_j(t)$ and $a_j(t)$ are optimized values chosen by individuals (given the initial assets $a_j(j)=0$). The per capita variables $c(t)$ and $a(t)$ are, instead, average values affected by the age structure of the population. This distinction turns out to be relevant when computing growth rates. Time-differentiation of (10) yields

$$\frac{\dot{c}(t)}{c(t)} + \frac{\dot{p}(t)}{p(t)} = r(t) - \rho - \underbrace{\frac{\psi(\rho+\delta)}{\gamma(1+\psi)} \cdot \frac{a(t)}{w(t)}}_{\text{Financial wealth dilution}} . \tag{11}$$

Comparing this expression to the individual Euler equation (3), we observe that the growth rates of per capita and individual consumption expenditure differ by the last term in (11), which represents the rate of *financial wealth dilution* induced by fertility – i.e., the share of per capita wealth that the members of each new cohort capture upon their arrival: from (10) and (6), we have

$$\underbrace{\frac{\psi(\rho+\delta)}{\gamma(1+\psi)} \cdot \frac{a(t)}{w(t)}}_{\text{Financial wealth dilution}} = \frac{A(t)/L(t)}{h(t)+A(t)/L(t)} \cdot \frac{B(t)}{L(t)}. \tag{12}$$

Financial wealth dilution affects per capita consumption growth because generations are disconnected. Since wealth is not redistributed in a dynastic fashion through intergenerational transfers, new cohorts enter the economy with zero assets and start pursuing their own accumulation and fertility plans independently from their predecessors. We might label this phenomenon as *passive* wealth dilution, in the sense that the consumption possibilities of each generation are subject to the accumulation and fertility decisions of all the subsequent generations. In fact, passive wealth dilution does not arise in models with pure altruism where the head of the dynasty optimizes the use of all private assets over an infinite time horizon, and the last term in (11) disappears. While these general characteristics of the wealth dilution mechanism have long been recognized in the literature – see the early contributions by Buiter (1988) and Weil (1989) – our analysis will add an important insight. In the present model, financial wealth dilution interacts with fertility choices and is thus both a consequence and a determinant of population dynamics. More precisely, financial wealth dilution tends to reduce the economy's fertility rate by limiting consumption expenditure, as we show next.

2.4 Fertility dynamics: expenditure and wage channels

To gain insight into population-fertility interactions, consider how the fertility rate b would respond to a change in population L for given levels of aggregate financial wealth A and individual human wealth h. From (6) and (10), the fertility rate equals

$$b(t) = \frac{\psi}{\gamma} \frac{\overbrace{p(t)\,c(t)}^{\text{Expenditure channel}}}{\underbrace{w(t)}_{\text{Wage channel}}} = \frac{\psi}{\gamma} \frac{\rho+\delta}{1+\psi} \left[\frac{A(t)}{L(t)} + h(t) \right] \frac{1}{w(t)} \tag{13}$$

The central term of (13) shows that for given levels of wealth, changes in population size affect the fertility rate through two channels. The *expenditure channel* incorporates the mechanism of wealth dilution discussed in the previous subsection: an increase in L for given A reduces assets per capita $a = A/L$, and thereby consumption expenditure per capita. Hence, a growing population tends to reduce fertility through the dilution of financial wealth. The *wage channel*, instead, operates through the impact of population on the wage that prevails on the labor market: population dynamics affecting the equilibrium wage rate will also affect fertility by modifying the households' opportunity cost of reproduction.

The impact of population on fertility via the wage channel is generally ambiguous: its sign and strength are determined on the supply side of the economy, which we have not modeled yet. The key equation is the growth rate of the fertility rate: time-differentiating (13), and substituting the Euler equation (11) for consumption growth along with the dynamic wealth constraint (7) in per capita terms, we obtain

$$\frac{\dot{b}(t)}{b(t)} = b(t) \left(1 + \gamma \frac{1+\psi}{\psi} \cdot \frac{w(t)}{a(t)} \right) - \rho - \delta - \frac{w(t)}{a(t)} + \frac{\dot{a}(t)}{a(t)} - \frac{\dot{w}(t)}{w(t)} - \frac{\psi(\rho+\delta)}{\gamma(1+\psi)} \cdot \frac{a(t)}{w(t)}, \tag{14}$$

where the last term is the rate of financial wealth dilution. Equation (14) delivers fundamental information because it incorporates the aggregation of all households' intertemporal decisions concerning *both* fertility and consumption choices into a single expression that only contains two variables, b and a/w, and their respective growth rates.[6] Therefore, by combining (14) with a model of the supply side determining the a/w ratio, we can characterize the equilibrium dynamics of the economy as a reduced system in three 'core variables': population, fertility, and the asset-wage ratio. In this respect, we stress that

[6] The behavior of households' propensities to consume out of wealth is incorporated in (13) and, hence, in (14).

different specifications of the supply side will deliver different predictions. The following taxonomy considers four general frameworks.

(i) *Models without financial wealth.* Suppose that financial assets do not exist because the only accumulable factor of production is labor. In this setup, agents cannot pursue consumption smoothing and equation (14) does not apply. Instead, the aggregate wealth constraint (7) yields that per capita consumption expenditure is proportional to the wage. From this follows immediately that the fertility rate, b, is constant and generally different from the death rate δ. This result is independent of the specifics of production and complies with Lucas' (2002) observation that labor-land models assuming a time-cost of reproduction typically predict that population grows exponentially at a constant rate.[7] For our purposes, the key point is that models without financial assets remove wealth dilution from the analysis.

(ii) *Neoclassical models with physical capital.* In these models financial assets are claims on physical capital and both labor and capital exhibit diminishing marginal returns with constant returns to scale. For given aggregate capital, an increase in population reduces capital per capita and thus has contrasting effects on the fertility rate: while the equilibrium wage falls – reducing the opportunity cost of child rearing – wealth dilution reduces consumption expenditure per capita. Most importantly, the endogenous accumulation of capital brings the economy towards a constant capital-labor ratio and, crucially, a constant asset-wage ratio a/w in the long run. From (14), this implies that the fertility rate becomes constant and population grows exponentially at a constant rate (proof in Appendix A). Hence, neoclassical models do not produce a wealth dilution mechanism that is sufficiently strong to stabilize the population.

(iii) *Endogenous growth models with constant returns to capital.* These models assume constant returns to the accumulable factor but keep the assumption of decreasing marginal returns to labor (e.g., Romer, 1986). Since the wage is inversely related to the population level, the effects of a larger population for given wealth are similar to those occurring in neoclassical models, but the implications for fertility are

[7] For example, assume that production combines labor with a fixed input land. These specifics yield an equilibrium wage that is decreasing in population due to diminishing returns but does not produce a feedback of population on fertility.

drastically different due to the strong scale effect – the property that the return to capital accumulation is increasing in population size. We characterize the dynamics for this model in Appendix A. The main result is that the model produces either explosive/degenerate paths, or a steady state with constant population and constant endogenous growth of variables per capita. Such a steady state, however, admits positive output growth only under very restrictive conditions on parameters and, most importantly, does not exist without assuming the strong scale effect.

(iv) *Endogenous growth models with costly R&D.* In R&D-based models of endogenous growth, financial wealth A represents the aggregate value of firms that raise productivity by accumulating intangible assets such as knowledge and ideas. For our purposes, we need to distinguish two sub-classes of models with radically different implications. The so called first-generation models (e.g., Romer, 1990; Aghion and Howitt, 1992) exhibit the strong scale effect and behave very much like the AK model discussed above. More interesting to us are the models with endogenous market structure (Peretto, 1998; Dinopoulos and Thompson, 1998, Howitt, 1999) that remove the strong scale effect by combining in-house R&D with horizontal innovations that create new firms. In particular, when both types of R&D compete for labor as an input, the wage response to increased population fully abstracts from neoclassical mechanisms of diminishing returns (Peretto and Connolly, 2007). These properties offer a radically different theory of population dynamics because, contrary to what we observe in frameworks (i)-(iii), the expenditure channel is fully operative and is neither offset nor dominated by a counteracting wage channel. We investigate this point in the remainder of our analysis by modelling the supply side of the economy according to Peretto and Connolly's (2007) specification.

The above considerations suggest two main remarks. First, frameworks (i)-(ii) neutralize the role of wealth dilution as the potential source of negative feedbacks of population on fertility and, more generally, exhibit a structural tendency to generate long-run equilibria where the fertility rate is constant.[8] The typical approach to rationalize declining and/or

[8] This is not a mere technical point: the economics literature on fertility makes extensive use of frameworks (i)-(ii) to address research questions – i.e., the income-fertility relationship and the determinants of the fertility decline – that ultimately require explaining how increases in the size of the population pull down subsequent fertility rates.

constant population in these frameworks is to include Malthusian mechanisms that create congestion in the use of essential natural resources and/or some form of quality-quantity trade-off for children – a prominent example is Galor and Weil (2000). While this approach is worthwhile and produces important insights, in this paper we deliberately take another route by excluding Malthusian mechanisms from the analysis. The second remark is that R&D-based models without scale effects can provide fundamentally different insights on population-fertility interactions because their core mechanism of wealth creation – i.e., the accumulation of intangible assets raising the mass of firms and each firm's profitability – propagates the wealth dilution mechanism and thereby gives rise to a general-equilibrium negative feedback of population on fertility. We formally investigate this point in the next two sections, obtaining a theory of the population level that fully abstracts from the Malthusian mechanisms emphasized in the existing literature.

3 The Production Side

The economy produces the final consumption good by means of differentiatied intermediates produced by monopolistic firms. Productivity growth is driven by both vertical and horizontal innovations in the intermediate sector: incumbents pursue vertical R&D to raise internal productivity, while outside entrepreneurs create new firms to enter the market. The model specifics draw on Peretto and Connolly (2007), which yields a transparent derivation of the equilibrium relationships linking the total value of firms to population size and to the market wage rate.

3.1 Final Sector

A competitive sector produces the final consumption good by assembling differentiated intermediate products according to the technology

$$C(t) = N(t)^{\chi - \frac{\epsilon}{\epsilon - 1}} \cdot \left(\int_0^{N(t)} x_i(t)^{\frac{\epsilon - 1}{\epsilon}} di \right)^{\frac{\epsilon}{\epsilon - 1}}, \tag{15}$$

where N is mass of intermediates, x_i is the quantity of the i-th intermediate good, $\epsilon > 1$ is the elasticity of substitution between pairs of intermediates, and $\chi > 1$ is the degree of increasing returns to specialization. Final producers take all prices and the mass of goods as given, and demand intermediate goods according to the profit-maximizing condition

$$p_{xi}(t) = \frac{p(t)\, C(t)}{\int_0^{N(t)} x_i(t)^{\frac{\epsilon-1}{\epsilon}} di} \cdot x_i(t)^{-\frac{1}{\epsilon}} \tag{16}$$

where p_{xi} is the price of intermediate good i.

3.2 Intermediate Producers: Incumbents

The typical intermediate firm produces according to the technology

$$x_i(t) = z_i(t)^{\theta} \cdot (\ell_{xi}(t) - \varphi), \tag{17}$$

where z_i is firm-specific knowledge, $\theta \in (0,1)$ is an elasticity parameter, ℓ_{xi} is labor employed in production, and $\varphi > 0$ is overhead labor. The firm accumulates knowledge according to

$$\dot{z}_i(t) = \omega Z(t) \cdot \ell_{zi}(t), \tag{18}$$

where ℓ_{zi} is labor employed in vertical R&D. The productivity of R&D employment is given by parameter $\omega > 0$ times $Z(t)$, a measure of the economy's stock of public knowledge defined as

$$Z(t) = \sigma(N(t)) \int_0^{N(t)} z_j(t)\, dj = \frac{1}{N(t)} \int_0^{N(t)} z_j(t)\, dj. \tag{19}$$

Expression (19) posits that public knowledge is a weighted sum of firm-specific stocks of knowledge z_j. The weight $\sigma(N)$ is a function of the mass of existing goods N to capture – in reduced form – features of the mechanism through which firms cross-fertilize each other: when a firm j develops a more efficient process to produce its own differentiated good, it also generates non-excludable knowledge which spills over into the public domain, and the extent to which this new knowledge is useful to firm $i \neq j$ depends on how far apart in technological space the differentiated products i and j are. The operator $\sigma(N) = 1/N$ is a simple way to formalize the idea that as the mass of goods increases, the *average* technological distance between existing goods increases as well. This in turn translates into weaker spillovers from any given stock of firm-specific knowledge.

In the intermediate sector, each firm faces a constant probability $\mu > 0$ of disappearing as a result of, e.g., product obsolescence.[9] Therefore, the incumbent monopolist at time

[9] Parameter μ is essentially the average death rate of intermediate firms. In the main text, we do not refer to μ as the firms' death rate in order to avoid confusion with the households' death rate δ.

t chooses the time paths $\{p_{xi}, x_i, \ell_{xi}, \ell_{zi}\}$ that maximize the present-value of the expected profit stream

$$V_i(t) = \int_t^{\infty} \left[p_{xi}(t) x_i(t) - w(t) \ell_{xi}(t) - w(t) \ell_{zi}(t)\right] e^{-\int_t^v (r(s)+\mu) ds} dv, \tag{20}$$

subject to the technologies (17)-(18) and the demand schedule (16). The solution to this problem yields the standard mark-up pricing rule (see Appendix B) and the dynamic no-arbitrage condition

$$r(t) = \left[\theta \cdot \frac{\epsilon - 1}{\epsilon} \cdot \frac{p_{xi}(t) x_i(t)}{z_i(t)} \cdot \frac{\omega Z(t)}{w(t)}\right] + \frac{\dot{w}(t)}{w(t)} - \frac{\dot{Z}(t)}{Z(t)} - \mu. \tag{21}$$

Expression (21) equates the market interest rate to the firm's rate of return from knowledge accumulation given by the right hand side, where the term in square brackets is the marginal profit from increasing firm's knowledge z_i. This is the key condition determining the equilibrium rate of vertical innovations in the economy.

3.3 Intermediate Producers: Entrants

Agents can allocate their labor time to developing new intermediate goods, designing the associated production processes, and setting up firms to serve the market. This process of horizontal innovation or, equivalently, entrepreneurship, increases the mass of firms, N, over time. At time t, an entrant, denoted i without loss of generality, correctly anticipates the value $V_i(t)$ that the new firm will create. Recalling that a constant fraction $\mu > 0$ of the existing firms disappears in each instant, the net increase in the mass of firms generated by entry is given by the technology

$$\dot{N}(t) = \eta \frac{N(t)}{L(t)^{\varkappa}} \ell_N(t) - \mu N(t), \qquad 0 \leqslant \varkappa < 1, \tag{22}$$

where ℓ_N is the total amount of labor invested by outside entrepreneurs in horizontal R&D. The productivity of labor in this activity depends on the exogenous parameter $\eta > 0$ and on two endogenous variables, the mass of firms and population size. The positive effect of the mass of firms, N, captures the intertemporal spillovers characteristic of the first-generation models of endogenous growth (Romer, 1990). The negative effect of population size, represented by the term $1/L^{\varkappa}$, captures the notion that entering large markets requires more effort (Peretto and Smulders, 2002): parameter $\varkappa$ regulates the intensity of this market-size effect and thereby the wage response to population change. All our results remain valid even if we exclude market-size effects from the analysis by setting $\varkappa = 0$.

The free-entry condition associated with technology (22) states that the market prices firms at their cost of creation:[10]

$$V_i(t) = \frac{w(t) L(t)^{\varkappa}}{\eta N(t)}. \tag{23}$$

This structure of the intermediates market implies that wealth creation has two dimensions. On the one hand, incumbent firms accumulate knowledge and drive their market valuation through this process. On the other hand, free-entry in the intermediate sector pins down the market valuation of firms from the cost side. Therefore, the market wage rate and the value of firms are determined by both types of R&D activities in equilibrium, as we show below.

3.4 Knowledge, Wage and Assets

The model exhibits a symmetric equilibrium where firms make identical decisions. The labor market clearing condition reads

$$\ell_X(t) + \ell_Z(t) + \ell_N(t) = L(t) - \gamma B(t), \tag{24}$$

where $\ell_X \equiv N\ell_{xi}$ and $\ell_Z \equiv N\ell_{zi}$ are aggregate employment levels in intermediates production and in vertical R&D, respectively, and the right-hand side is total labor supply. Combining (24) with the profit-maximizing conditions of intermediate producers, we obtain the equilibrium real wage

$$\frac{w(t)}{p(t)} = \frac{\epsilon - 1}{\epsilon} Z(t)^{\theta} N(t)^{\chi - 1}. \tag{25}$$

Expression (25) shows that real wage growth hinges on both types of R&D, namely, vertical innovations that raise public knowledge Z and horizontal innovations that increase the mass of firms N. Considering the aggregate value of firms, the equilibrium in the financial market requires $A = NV$ so that the free-entry condition (22) yields

$$A(t) = N(t) V(t) = \frac{w(t) L(t)^{\varkappa}}{\eta}. \tag{26}$$

Combining (26) with (25), we can write *real* aggregate wealth in terms of its fundamental determinants, namely, the stock of public knowledge, the mass of firms, and population:

$$\frac{A(t)}{p(t)} = \frac{\epsilon - 1}{\epsilon \eta L(t)^{\varkappa}} \cdot Z(t)^{\theta} N(t)^{\chi - 1}. \tag{27}$$

[10] Given the entry technology (22), the free entry condition (23) establishes that the total value of new firms, $\int_0^{\dot{N}+\mu N} V_i di$, matches the total cost of their creation, $w\ell_N$.

From (27), the economy's rate of wealth creation depends on both vertical and horizontal innovation rates, $\dot{Z}/Z$ and $\dot{N}/N$, as well as on population growth via market-size effects on labor productivity. The next section exploits these results to characterize the economy's general equilibrium.

4 General Equilibrium

This section merges the demographic block of the model (section 2) with the supply side (section 3), and characterizes the resulting equilibrium dynamics. We show that the combined mechanisms of wealth creation and wealth dilution generate a steady state in which a constant endogenous population level coexists with sustained endogenous output growth. In the remainder of the analysis, we take the final good as our numeraire and set $p(t) = 1$.

4.1 The Dynamic System

Our previous discussion of intertemporal choices (subsect. 2.4) showed that the model's core dynamics take the form of a reduced system whereby the demographic law (5), the fertility equation (14), and the supply side of the economy determine the time paths of L, b, and a/w. The key ingredient coming from the supply side is the equilibrium relationship (26), which links the total value of firms to labor productivity in horizontal innovations. Dividing both sides of (26) by population size, we obtain

$$\frac{a(t)}{w(t)} = \frac{1}{\eta L(t)^{1-\varkappa}}. \tag{28}$$

Equation (28) establishes that the equilibrium value of the asset-wage ratio is strictly decreasing in the population level, even when market-size effects are ruled out by setting $\varkappa = 0$. The interpretation is that along the equilibrium path, increased population reduces financial wealth per capita without generating an equivalent decline in the market wage rate. The negative relationship between a/w and L is a distinctive feature of our model[11] and bears crucial implications for fertility dynamics because, from (13), the fertility rate

[11] The negative relationship between a/w and L in (28) essentially says that when population grows, the wealth dilution effect originating in the demographic structure dominates because the assumed supply-side structure does not generate an offsetting (negative) wage response to increased population. This result thus hinges on combining the Blanchard-Yaari demographic structure with the endogenous growth model with simultaneous innovations (Peretto, 1998).

is positively related to assets per capita and negatively related to the wage rate. In fact, condition (28) turns out to be essential to obtain a *negative feedback* of population on fertility along the equilibrium path and, hence, to produce a theory of finite population in the absence of physical constraints.

Rewriting the demographic law (5) in terms of net fertility, and using (28) to substitute a/w in the fertility equation (14), we obtain the autonomous dynamic system

$$\frac{\dot{L}(t)}{L(t)} = b(t) - \delta, \tag{29}$$

$$\frac{\dot{b}(t)}{b(t)} = \frac{\gamma(1+\psi)\,b(t)-\psi}{\psi}\eta L(t)^{1-\varkappa} - \rho + \varkappa\left(b(t)-\delta\right) - \frac{\psi(\rho+\delta)}{\gamma(1+\psi)}\frac{1}{\eta L(t)^{1-\varkappa}}. \tag{30}$$

Equation (30) delivers a complete picture of the feedback effects of population on fertility along the equilibrium path: it encompasses all households' intertemporal decisions concerning consumption and fertility and incorporates the equilibrium relationship (28). Increased population reduces assets per worker relative to the wage rate, a/w, and this affects fertility via financial wealth dilution – i.e., the last term in (30) – and via changes in the rate of return to assets, which modifies the agents' consumption possibilities and their willingness to rear children.

System (29)-(30) fully determines the dynamics of population and fertility rates. The stationary loci are

$$\dot{L} = 0 \quad \Rightarrow \quad b = \delta, \tag{31}$$

$$\dot{b} = 0 \quad \Rightarrow \quad \bar{b}(L) = \frac{\varkappa\delta + \eta L^{1-\varkappa}}{\varkappa + \gamma\frac{1+\psi}{\psi}\eta L^{1-\varkappa}} + \frac{\rho\eta L^{1-\varkappa} + \frac{\psi}{\gamma}\frac{\rho+\delta}{1+\psi}}{\varkappa\eta L^{1-\varkappa} + \gamma\frac{1+\psi}{\psi}\left(\eta L^{1-\varkappa}\right)^2}. \tag{32}$$

The $\dot{L} = 0$ locus establishes that population is stationary when the gross fertility rate matches the mortality rate. The $\dot{b} = 0$ locus is a negative relationship between fertility and population, $\bar{b}(L)$, displaying the properties (see Appendix C)

$$\partial\bar{b}(L)/\partial L < 0, \qquad \lim_{L\to 0}\bar{b}(L) = +\infty, \qquad \lim_{L\to\infty}\bar{b}(L) = \frac{\psi}{\gamma(1+\psi)}. \tag{33}$$

The asymptotic properties of the $\dot{b} = 0$ locus imply that the dynamic system admits a simultaneous steady state (L_{ss}, b_{ss}) in which fertility is at replacement level and population is constant. This means that the negative feedbacks of population size on fertility – stemming from wealth dilution and propagated by relationship (28) – can drive net fertility to zero. The phase diagram in Figure 1, graph (a), clarifies that such steady state exists when the $\dot{L} = 0$ locus lies strictly above the horizontal asymptote of the $\dot{b} = 0$ locus, given by the second limit appearing in (33). Consequently, the steady state (L_{ss}, b_{ss}) exists and is unique if parameters satisfy the condition

$$\gamma\delta\,(1+\psi) > \psi. \tag{34}$$

The intuition behind (34) is that the negative feedback of population on fertility brings population growth to a halt when the marginal cost of child-bearing γ is high relative to the preference for children ψ, given the probability of death, δ.

The steady state with constant population is the focus of our analysis. We nonetheless briefly discuss two cases in which the steady state does not exist. First, when parameters are such that $\gamma\delta\,(1+\psi) \leqslant \psi$, inequality (34) is violated and the stationary loci (31)-(32) do not exhibit any intersection: when mortality and reproduction costs are too low, the negative effect of population on fertility does not suffice to stabilize population. In this case, the economy converges asymptotically to a constant fertility rate, $\lim_{L\to\infty} \bar{b}\,(L) = \gamma\,(1+\psi)\,/\psi$, which strictly exceeds the mortality rate and thus yields a perpetually growing population. A somewhat similar behavior arises when inequality (34) is satisfied but $\varkappa \to 1$, which implies that the wealth-wage ratio a/w becomes independent of population in equation (28). In this special case, equation (30) reproduces the well-known result of exponential growth: the fertility rate jumps to a constant value which can deliver a perpetually growing or shrinking population.[12] The reason is that when $\varkappa \to 1$, the model lacks any feedback from population size to fertility.

4.2 The Steady State with Constant Population

When the steady state (L_{ss}, b_{ss}) exists, the model delivers a theory of the *population level*. The phase diagram in Figure 1, graph (a), shows that given the initial population $L\,(0)$, the economy jumps onto the saddle path by selecting initial fertility $b\,(0)$, and then converges to the steady state.[13] The trajectory that starts from $L\,(0) < L_{ss}$ represents the case that

[12] With $\varkappa = 1$, the $\dot{b} = 0$ locus reduces to the constant fertility rate $b_{(\varkappa=1)} \equiv \frac{\psi\eta}{\psi+\gamma\eta(1+\psi)} + \frac{\rho+\delta}{\gamma\eta(1+\psi)}$, which may be above or below the mortality rate depending on parameter values.

[13] The diverging trajectories in Figure 1, graph (a), can be ruled out as equilibrium paths by standard arguments (i.e., they would imply explosive dynamics in $b\,(t)$ that violate the necessary conditions for utility maximization).

is empirically relevant for most developed countries: population grows during the transition but the fertility rate declines and eventually becomes equal to the mortality rate δ. The following proposition formalizes the result.

Proposition 1 *If parameters satisfy $\gamma\delta(1+\psi) > \psi$, the steady state (L_{ss}, b_{ss}) is saddle-point stable and represents the long-run equilibrium of the economy:*

$$\lim_{t\to\infty} b(t) = b_{ss} \equiv \delta \tag{35}$$

$$\lim_{t\to\infty} L(t) = L_{ss} \equiv \left[\frac{\psi}{\eta 2} \cdot \frac{\rho + \sqrt{\rho^2 + 4(\rho+\delta)\left(\delta - \frac{\psi}{\gamma(1+\psi)}\right)}}{\gamma(1+\psi)\delta - \psi}\right]^{\frac{1}{1-\varkappa}} \tag{36}$$

$$\lim_{t\to\infty} \frac{a(t)}{w(t)} = \left(\frac{a}{w}\right)_{ss} \equiv \frac{1}{\eta L_{ss}^{1-\varkappa}} = \frac{2}{\psi} \cdot \frac{\gamma(1+\psi)\delta - \psi}{\rho + \sqrt{\rho^2 + 4(\rho+\delta)\left(\delta - \frac{\psi}{\gamma(1+\psi)}\right)}} \tag{37}$$

The most striking result contained in Proposition 1 is that, in the long run, the ratio between assets per worker and the wage rate is exclusively determined by demographic factors and preferences: expression (37) shows that a/w converges to a steady state level $(a/w)_{ss}$ that does not depend on technology parameters. Nonetheless, the entry technology (22) affects steady-state population: from (36), the long-run level L_{ss} is linked to horizontal R&D through the parameters η and $\varkappa$. The reason for these results is that the dominant feedback of population on fertility comes from financial wealth dilution, which originates in the economy's demographic structure. While the supply-side structure guarantees that the overall feedback of population on fertility is indeed negative in equilibrium, the core determinant of the steady state (L_{ss}, b_{ss}) remains the wealth dilution mechanism:[14] households keep on adjusting fertility rates until they achieve a specific ratio $(a/w)_{ss}$ that stabilizes their marginal rate of substitution between consumption and child-rearing, at which point the economy is in the steady state. Although the specific level $(a/w)_{ss}$ is pinned down by demographic and preference parameters, population in the long run still depends on technology because the steady-state level L_{ss} that is compatible with $(a/w)_{ss}$ depends on the response of the wage rate to population size.

[14] The fact that the core mechanism stabilizing population is wealth dilution is confirmed by condition (34), which establishes that the existence of the steady state (L_{ss}, b_{ss}) only depends on demographic and preference parameters, $(\delta, \gamma, \psi, \rho)$.

With respect to the transitional dynamics, three remarks are in order. First, the dynamic system (29)-(30) also determines the equilibrium path of the *consumption-assets* ratio. Combining (13) with (28), we obtain

$$\frac{C(t)}{A(t)} = \frac{\gamma}{\psi} \cdot \frac{b(t)\, w(t)}{a(t)} = \frac{\gamma}{\psi} \cdot b(t)\, \eta L(t)^{1-\varkappa}\,. \tag{38}$$

In the long run, the consumption-assets ratio converges to the steady state level

$$\lim_{t\to\infty} \frac{C(t)}{A(t)} = \left(\frac{C}{A}\right)_{ss} = \frac{\gamma}{\psi} \cdot \frac{b_{ss}}{(a/w)_{ss}} \tag{39}$$

which, by Proposition 1, depends exclusively on demography and preference parameters. The property that demographic forces determine both a/w and C/A has consequences that go well beyond the questions typically addressed by fertility models: the functional income distribution is strongly driven by demographic factors, a prediction in sharp contrast with traditional – e.g., neoclassical – models of macroeconomic growth. We will address this point in section 5.

The second remark relates to the transitional co-movements of fertility and consumption. The time path of $C(t)/A(t)$ is not necessarily monotonous because $b(t)$ and $L(t)$ move in opposite directions over time. However, it follows from (38) that the ratio between consumption *per capita* and *total* assets does exhibit monotonous dynamics because $c(t)/A(t)$ is positively related to fertility and negatively related to population along the equilibrium path. In particular, starting from $L(0) < L_{ss}$, the transition features declining fertility rates accompanied by positive population growth and a declining c/A ratio over time. These equilibrium co-movements are empirically plausible and we show in Appendix C that the shape of the saddle path is indeed consistent with panel data for OECD economies: both the negative fertility-population relationship and the inverse relationship between c/A and population are strong and significant.

The third remark relates to the nature of the steady state (L_{ss}, b_{ss}). Equation (36) says that long-run population depends on preference parameters, fertility costs and the productivity of labor in creating new firms. It does not depend on fixed endowments as we purposefully omitted them from the model. In other words, the steady state (L_{ss}, b_{ss}) is *non-Malthusian*: the fact that population converges to a finite level is not due to binding physical constraints. To the best of our knowledge, this is a novel result: in the existing literature, a finite endogenous population *level* is typically the outcome of Malthusian mechanisms whereby finite natural resource endowments impose limits on population size. Our model, instead, delivers a theory of the population level in which net fertility approaches zero because the negative feedback of population on fertility originates in the dilution of *financial* wealth.

4.3 Wealth Creation and Output Growth

The economy's rate of wealth creation depends on both horizontal and vertical innovations. From (27), the growth rate of total assets equals

$$\frac{\dot{A}(t)}{A(t)} = \theta\frac{\dot{Z}(t)}{Z(t)} + (\chi - 1)\frac{\dot{N}(t)}{N(t)} - \varkappa\frac{\dot{L}(t)}{L(t)}. \tag{40}$$

Provided that certain restrictions hold, both types of R&D activities are operative along the equilibrium path: the remainder of the analysis assumes that such restrictions hold so that employment in both activities is strictly positive (see Appendix C for details). Horizontal and vertical innovation rates interact according to the dynamic equations[15]

$$\frac{\dot{Z}(t)}{Z(t)} = (1-\gamma b(t))\frac{w(t)}{a(t)} + \left[\left(\frac{\epsilon-1}{\epsilon}\right)\frac{\omega\theta}{\eta}\frac{L(t)^{\varkappa}}{N(t)} - 1\right]\frac{c(t)}{a(t)} - \varkappa\frac{\dot{L}(t)}{L(t)} - \mu, \tag{41}$$

$$\frac{\dot{N}(t)}{N(t)} = (1-\gamma b(t))\frac{w(t)}{a(t)} - \frac{\epsilon-1}{\epsilon L(t)^{2\varkappa}}\frac{c(t)}{a(t)} - \mu - \left[\frac{\eta}{L(t)^{\varkappa}}\left(\varphi + \frac{1}{\omega}\frac{\dot{Z}(t)}{Z(t)}\right)\right]\cdot N(t), \tag{42}$$

where the time paths of w/a and c/a are fully determined by the dynamic system studied in the previous subsection. The central message of (41)-(42) is that the growth rates of knowledge and of the mass of firms exhibit negative co-movement over time. While the entry of new firms reduces the profitability of each individual firm's investment in knowledge through market fragmentation, investment in knowledge slows down entry by diverting labor away from horizontal R&D activity.[16] Importantly, these co-movements bring the economy towards a long-run equilibrium in which vertical R&D generates sustained knowledge growth whereas the mass of firms converges to a constant level in the long run:

[15] Equation (41) follows from aggregating the private return to knowledge investment (21) across firms, and (42) derives from the entry technology (22) and the labor market clearing condition (24).

[16] The market-fragmentation effect is captured by the term in square brackets in (41) whereby an increase in N reduces $\dot{Z}/Z$ by squeezing the marginal profit that each firm gains from investing in own knowledge. The labor-reallocation mechanism that negatively affects horizontal R&D is captured by the last term in (42).

Proposition 2 *In the steady state (L_{ss}, b_{ss}), the mass of firms is constant and finite, $\lim_{t\to\infty} N(t) = N_{ss} > 0$. During the transition, the mass of firms follows a logistic process of the form*

$$\frac{\dot{N}(t)}{N(t)} = q_1(b(t), L(t)) - q_2(b(t), L(t)) \cdot N(t), \tag{43}$$

where $q_1(b, L)$ and $q_2(b, L)$ converge to finite constants, $q_1(b_{ss}, L_{ss}) > 0$ and $q_2(b_{ss}, L_{ss}) > 0$, in the long run. With operative vertical R&D, the long-run mass of firms equals

$$\lim_{t\to\infty} N(t) = N_{ss} \equiv \frac{\eta L_{ss}^{1-\varkappa}\left[1 - \gamma b_{ss} - \left(\theta + \frac{1}{L_{ss}^{2\varkappa}}\right)\frac{\epsilon-1}{\epsilon}\frac{\gamma}{\psi} b_{ss}\right] - \mu}{\varphi - \frac{1}{\omega}\left[\frac{(1+\psi)\gamma b_{ss} - \psi}{\psi}\eta L_{ss}^{1-\varkappa} + \mu\right]} \cdot \frac{L_{ss}^{\varkappa}}{\eta} > 0 \tag{44}$$

which exhibits $dN_{ss}/dL_{ss} > 0$ for any $\varkappa \in [0, 1)$.

Proposition 2 establishes that the process of firms' entry eventually stops in the long run, a general result that holds regardless of whether vertical R&D is operative. The intuition is that outside entrepreneurs create new firms as long as their anticipated market share yields the desired rate of return but, as new firms join the intermediate sector, each firm's market share declines: the profitability of entry eventually vanishes due to the competing use of labor in the production of intermediates – which is subject to the fixed operating cost $\varphi > 0$ – and in vertical R&D activities if operative.[17] When the mass of firms approaches the steady state value N_{ss}, further product creation is not profitable given total labor availability and total consumption expenditure – that is, the profitability of entry fully adjusts to the endogenous values (b_{ss}, L_{ss}) in the long run. This is why the long-run mass of firms N_{ss} is positively related to population size L_{ss}.

Since population and the mass of firms are asymptotically constant, the only source of productivity growth in the long run is vertical R&D. From (41), the growth rate of knowledge is

$$\lim_{t\to\infty} \frac{\dot{Z}(t)}{Z(t)} = g_Z^{ss} \equiv \frac{\epsilon - 1}{\epsilon} \cdot \frac{\omega\theta}{\eta}\frac{L_{ss}^{\varkappa}}{N_{ss}}\left(\frac{c}{a}\right)_{ss} + \underbrace{(1 - \gamma b_{ss})\left(\frac{w}{a}\right)_{ss} - \left(\frac{c}{a}\right)_{ss} - \mu}_{\frac{\dot{A}(t)}{A(t)} - r(t) - \mu}, \tag{45}$$

which is strictly positive as long as the mass of firms N_{ss} is not too large. In the right hand side of (45), the first term captures an intra-termporal gain, namely, the increase in

[17] In the logistic process (43), the term $q_1(b, L)$ represents the incentive to create a new firm, given by the market share anticipated by individual entrepreneurs, whereas $q_2(b, L)$ measures the decreased profitability of entry induced by market crowding. See Appendix C for detailed derivations.

firms' profitability given by a marginal increase in knowledge, which depends on the ratio between output sales and firms' value and is thus positively related to (c/a). The second and third terms capture, instead, the inter-temporal *net* gains from R&D investment given by the gap between wealth creation, $\dot{A}/A$, and the effective rate of firms' proft discount, $r+\mu$.

The economy's rate of wealth creation obeys equation (40). Since the mass of firms is asymptotically constant, $\dot{N}/N \to 0$, the growth rate of assets in the long run is proportional to that of knowledge, $\dot{A}/A \to \theta \cdot g_Z^{ss}$, and the same growth rate applies to final output in view of stationarity of the consumption-wealth ratio. The economy's long-run growth rate thus equals[18]

$$\lim_{t\to\infty}\frac{\dot{A}\left(t\right)}{A\left(t\right)}=\lim_{t\to\infty}\frac{\dot{C}\left(t\right)}{C\left(t\right)}=\theta g_Z^{ss}=\theta\left[1-\gamma b_{ss}+\left(\omega\theta\frac{\epsilon-1}{\eta\epsilon}L_{ss}^{\varkappa}-N_{ss}\right)\frac{\gamma}{\psi}\frac{b_{ss}}{N_{ss}}\right]\eta L_{ss}^{1-\varkappa}-\theta\mu. \tag{46}$$

Expression (46) shows that both technology and demography affect the pace of knowledge accumulation and, hence, economic growth in the long run. In particular, demography affects economic growth by modifying the composition of R&D investment: a higher steady-state population L_{ss} tends to boost horizontal innovations, yielding a larger mass of firms N_{ss} in the steady state (see Proposition 2). This mechanism plays a central role in determining the welfare consequences of demographic shocks, a point that we address in the quantitative analysis of section 6.

5 Demographic Shocks, Income Shares and Migration

Our theory delivers predictions that are in stark contrast with most traditional growth models. In the long run, the ratios of key macroeconomic variables – labor incomes, consumption and assets – are exclusively determined by demography and preference parameters. Therefore, exogenous demographic change – e.g., shocks on reproduction costs, life expectancy, or migration – has a first-order impact on the functional distribution of income, accumulation decisions and long-term economic growth. This section discusses these and related results by extending the model to include migration.

[18] The last term in (46) is obtained from (45) by substituting $(w/a)_{ss}$ and $(c/a)_{ss}$ with the steady-state values reported in (37) and (39).

5.1 Exogenous Shocks

The following Proposition summarizes the effects of exogenous shocks affecting the time cost of reproduction, the time preference rate, and the probability of death.

Proposition 3 *Exogenous increases in γ, ρ, and δ modify steady-state values as follows:*

$$\begin{aligned} db_{ss}/d\gamma &= 0 \qquad \text{and} \qquad dL_{ss}/d\gamma < 0, \\ db_{ss}/d\rho &= 0 \qquad \text{and} \qquad dL_{ss}/d\rho > 0, \\ db_{ss}/d\delta &> 0 \qquad \text{and} \qquad dL_{ss}/d\delta < 0, \end{aligned}$$

Figure 1 describes the above results in three phase diagrams where the economy is initially in the steady state (L_{ss}^o, b_{ss}^o) and then moves towards the after-shock steady state (L'_{ss}, b'_{ss}). An exogenous increase in γ reduces the long-run population level but does not affect steady-state fertility: while higher reproduction costs prompt workers to have fewer children during the whole transition to the new steady state, the fertility rate b_{ss} reverts towards its pre-shock level δ because the population decline increases consumption per worker via reduced wealth dilution. Considering changes in time preference, an increase in ρ prompts households to raise their propensity to consume out of wealth and enjoy higher consumption and fertility at earlier dates over the life-cycle. This 'discounting effect' yields higher fertility during the transition to the new steady state, which results in a larger population L_{ss} in the long run.

Considering changes in the probability of death, the result $\mathrm{d}L_{ss}/\mathrm{d}\delta < 0$ arises from two contrasting effects that deserve attention. On the one hand, a higher δ affects intertemporal choices in the same way as a higher time-preference rate in the consumption function (10): taken alone, this *discounting effect* of δ would tend to increase L_{ss} via the same mechanism generated by an increase in ρ. On the other hand, a higher death probability increases the economy's mortality rate, driving down population growth: this *mortality effect* of δ tends to reduce L_{ss} and increase b_{ss} because the fertility rate must compensate, in the long run, a faster population turnover. In the proof of Proposition 3, we establish that the mortality effect always dominates the discounting effect so that a higher probability of death reduces population in the long run, $\mathrm{d}L_{ss}/\mathrm{d}\delta < 0$. In Figure 1, graph (d), the upward shift in the $\dot{b} = 0$ locus represents the discounting effect whereas the upward shift in the $\dot{L} = 0$ locus represents the mortality effect. The initial and final steady states, respectively denoted by (L_{ss}^{o}, b_{ss}^{o}) and (L_{ss}', b_{ss}'), can be immediately compared to those generated by the time-preference shock described in graph (c). The fact that shocks on ρ and shocks on δ generate opposite effects on population size is relevant for assessing the impact of shocks on life expectancy, as we discuss below.

5.2 The 'key ratios': labor incomes, consumption and assets

In the steady state (L_{ss}, b_{ss}), the determination of crucial macroeconomic variables is qualitatively different from that suggested by traditional growth models. A useful benchmark for comparison is Blanchard's (1985) model, which combines Yaari's (1965) demographic structure with a standard neoclassical supply side: financial wealth consists of physical capital displaying decreasing marginal returns, and aggregate output is a linearly homogeneous function of capital and labor. In this framework, population grows exponentially at the same rate as that of accumulable inputs in the long run. This traditional notion of balanced growth, which dates back to Solow (1956), holds even in related models where fertility is endogenously determined by private choices (e.g., Becker and Barro, 1988). We can summarize the main differences between our predictions and the traditional ones as follows. Consider the long-run values of three endogenous variables that are of direct interest for growth analysis: the ratio of total labor incomes to total assets, the consumption-assets ratio, and the ratio of total labor incomes to consumption. In the current notation, these variables read

$$\frac{w(t)\,L(t)\,(1-\gamma b(t))}{A(t)}, \qquad \frac{C(t)}{A(t)}, \qquad \frac{w(t)\,L(t)\,(1-\gamma b(t))}{C(t)},$$

respectively. Both the traditional framework and our model predict that these 'key ratios' are stationary in the long run but the underlying mechanisms are different. Traditional balanced growth hinges on a stable input ratio in the long run: as the growth rate of capital adjusts to that of labor, financial wealth grows at the same rate as labor incomes while population grows forever at a constant rate. As consumption growth adjusts to the growth rate of inputs, the consumption-asset ratio and the labor share of national income are also stabilized in the long run. Instead, in the long-run equilibrium of our model, labor supply is constant but the wage rate $w(t)$ grows at the same rate as assets $A(t)$ in view of the free-entry condition (26): when $L(t) = L_{ss}$, the value of firms becomes proportional to unit labor costs, determining a stable ratio between total labor incomes and financial assets.

The departure from the predictions of traditional models is substantial, as emphasized in the next Proposition:

Proposition 4 *In the steady state* (L_{ss}, b_{ss})*, the ratios*

$$\begin{aligned}
\lim_{t\to\infty} \frac{w(t)\, L(t)\,(1-\gamma b(t))}{A(t)} &= \frac{1-\gamma\delta}{(a/w)_{ss}}, \\
\lim_{t\to\infty} \frac{C(t)}{A(t)} &= \frac{\gamma}{\psi}\cdot\frac{\delta}{(a/w)_{ss}}, \\
\lim_{t\to\infty} \frac{w(t)\, L(t)\,(1-\gamma b(t))}{C(t)} &= \psi\cdot\frac{1-\gamma\delta}{\gamma\delta},
\end{aligned}$$

are exclusively determined by demographic and preference parameters, with $(a/w)_{ss}$ *given by (37).*

Proposition 4 establishes that, in the long run, all the key ratios are exclusively determined by demography and preferences, a result in stark contrast with the predictions of traditional – in particular, neoclassical – models where such steady-state values are crucially, if not exclusively determined by technology. In our theory, exogenous shocks hitting demographic or preference parameters – and by extension, public policies directly affecting reproduction costs or life expectancy – have a first-order impact on the functional distribution of income, individual welfare, and economic growth. We quantitatively assess these effects in section 6.

5.3 Life Expectancy and Time Preference

A specific implication of Proposition 4 concerns the impact of life expectancy on labor incomes. In our model, exogenous shocks on the probability of death affect the wage-wealth ratio in the opposite direction as shocks on the time preference (see Appendix D):

$$\frac{\mathrm{d}}{\mathrm{d}\rho}\lim_{t\to\infty}\frac{w(t)}{a(t)} > 0 \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}\delta}\lim_{t\to\infty}\frac{w(t)}{a(t)} < 0. \tag{47}$$

Result (47) contrasts with the predictions of Blanchard's (1985) model where shocks affecting δ and ρ modify the steady state in qualitatively the same way. The established argument is that the probability of death acts primarily as an additional term of utility discounting: since the households' effective discount rate is $(\rho+\delta)$, shocks on the death probability are assimilated to shocks on impatience. In particular, Blanchard's (1985) model predicts that increases in ρ and δ unambiguously raise the wage-wealth ratio (see Appendix D):

$$\text{(Blanchard, 1985):} \qquad \frac{\mathrm{d}}{\mathrm{d}\rho} \lim_{t\to\infty} \frac{w(t)}{a(t)} > 0 \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}\delta} \lim_{t\to\infty} \frac{w(t)}{a(t)} > 0. \tag{48}$$

The intuition for result (48) is that, in a capital-labor economy, a higher effective discount rate prompts households to anticipate consumption and reduce capital accumulation; in the long run, a lower capital-to-labor ratio makes total labor incomes higher relative to aggregate capital. Therefore, in Blanchard's (1985) model, increases in δ modify the allocation via *discounting effects*. Our result (47) is different because in our model, the discounting effect is always dominated by the *mortality effect* (cf. subsection 5.1): a higher δ reduces the long-run population level L_{ss} and drives down total labor incomes via reductions in the work force that are not compensated by a proportional increase in wages. The negative impact of δ on the ratio wL/A is thus explained by the fact that changes in ρ and δ affect the long-run population level in opposite directions, as established in Proposition 3. This result is further expanded in section 6.2 by showing that the mass of firms and long-run growth rates also move in opposite directions in response to the two shocks.

5.4 Migration

Introducing migration is a natural extension of this model. On the one hand, as noted by Weil (1989), immigrants are by definition 'disconnected' generations and thus directly reinforce wealth dilution. On the other hand, net inflows of people also affect wealth creation because, in our model, a higher steady-state mass of workers L_{ss} boosts horizontal innovations and results in a larger mass of firms N_{ss} in the long run. We assess these mechanisms both analytically and quantitatively by making two assumptions that preserve the model's tractability. First, migrants enter or leave the economy exclusively at the beginning of their working age. Second, immigrants have the same preferences and life expectancy as domestic residents.[19]

In the remainder of the analysis, $B(t)$ represents domestic births and the new variable $M(t)$ denotes net inflows of migrants in the economy at instant t. The size at time j of the cohort 'entering the economy' at time j thus equals $k(j,j) = B(j) + M(j)$. The model is easily amended by modifying only a few equations in section 2 and in subsections 4.1-4.2. The necessary modifications can be summarized in three steps (see Appendix D for a

[19] The role of these two hypotheses is merely that of avoiding that migration introduce heterogeneities in preferences or in the age-composition of the population.

detailed discussion). First, the demographic law now includes net immigration: equation (5) is replaced by

$$\dot{L}(t) = B(t) + M(t) - \delta L(t). \tag{5'}$$

Second, net inflows of immigrant workers boost financial wealth dilution: the arrival of further disconnected generations, in addition to domestic births, directly affects the growth rate of consumption per capita and thereby the dynamics of the fertility rate through the dilution channel. Formally, the rate of financial wealth dilution appearing in (11), (12) and (14) is replaced by the augmented rate

$$\underbrace{\frac{\psi(\rho+\delta)}{\gamma(1+\psi)} \cdot \frac{B(t)+M(t)}{B(t)} \cdot \frac{a(t)}{w(t)}}_{\text{Augmented financial wealth dilution}} = \frac{A(t)/L(t)}{h(t)+A(t)/L(t)} \cdot \frac{B(t)+M(t)}{L(t)}. \tag{12'}$$

Third, migration modifies the dynamic system (29)-(30) and potentially its properties depending on how we specify the behavior of total inflows, $M(t)$, or alternatively, of the *net immigration rate* defined as $m(t) \equiv M(t)/L(t)$. Given our focus on exogenous inflows, we consider two basic alternatives: a constant level of net inflows, $M(t) = \bar{M}$, or a constant net immigration rate $m(t) = \bar{m}$.[20] In the first case, a constant flow of immigrants $\bar{M}$ makes the migration rate $m(t)$ generally time-varying and subject to the dynamics of total population. In the second case, a constant migration rate $\bar{m}$ implies a time-varying number of immigrants instead. Which specification is more suitable depends on the purpose of the analysis. In section 6, we perform numerical simulations assuming $M(t) = \bar{M}$ in order to assess the effects of immigration barriers – i.e., exogenous restrictions to inflows where the policy-target variable is the total number of immigrants. Nonetheless, both specifications of migration flows preserve our main conclusions and expand our notion of non-Malthusian steady state. In Appendix D, we modify the dynamic system (29)-(30) to include migration and we prove the following

Proposition 5 *(Steady state with migration) Assuming either $M(t) = \bar{M}$ or $m(t) = \bar{m}$, the equilibrium dynamics of $(L(t), b(t), m(t))$ exhibit a stable steady state (L_{ss}, b_{ss}, m_{ss})*

[20] The literature on forecasting models that incorporate demographic projections suggests that either $M(t)$ or $m(t)$ should follow mean-reverting functions of time. For example, Faruqee (2002) incorporates demographic projections in the Blanchard-Yaari model by means of calibrated trigonometric functions.

where

$$\lim_{t\to\infty} b(t) = b_{ss} \equiv \delta - m_{ss}, \tag{35'}$$

$$\lim_{t\to\infty} L(t) = L_{ss} \equiv \left[\frac{\psi}{\eta 2} \cdot \frac{\rho + \sqrt{\rho^2 + 4\delta(\rho+\delta)\left(1 - \frac{\psi}{\gamma(1+\psi)(\delta - m_{ss})}\right)}}{\gamma(1+\psi)(\delta - m_{ss}) - \psi} \right]^{\frac{1}{1-\varkappa}}, \tag{36'}$$

Such steady state exists provided that $\gamma(\delta - m_{ss})(1+\psi) > \psi$.

The nature of the long-run migration rate, $\lim_{t\to\infty} m(t) = m_{ss}$, depends on how the immigration process is specified. Assuming $m(t) = \bar{m}$, the migration rate is exogenous and our previous analysis of demographic shocks is virtually unchanged. Assuming $M(t) = \bar{M}$, the migration rate is endogenous and demographic shocks have slightly richer effects – relative to those described in Proposition 3 – because changes in steady-state population L_{ss} also induce changes in fertility b_{ss} via the migration rate $m_{ss} = \bar{M}/L_{ss}$. Aside from these second-order effects, both specifications of migration flows yield the same general insights. The first and most evident is that the fertility rate b_{ss} adjusts to the turnover rate $\delta - m_{ss}$ in the long run and is therefore negatively related to the (asymptotic) immigration rate.[21] The relevant consequence is that the immigration rate becomes a determinant of the 'key ratios' previously discussed (cf. subsection 5.2): as we show in Appendix D, all the expressions appearing in Proposition 4 will hold but with δ being replaced by $\delta - m_{ss}$. In particular, the ratio of labor incomes to consumption equals

$$\lim_{t\to\infty} \frac{w(t)L(t)(1-\gamma b(t))}{C(t)} = \psi \cdot \frac{1-\gamma(\delta - m_{ss})}{\gamma(\delta - m_{ss})}, \tag{49}$$

so that the total wage bill relative to consumption is *strictly increasing* in the net immigration rate. This result will be crucial for understanding the welfare consequences of immigration barriers in the quantitative analysis presented below.

6 Quantitative analysis

In this section, we calibrate the model to replicate the average values of key macroeconomic variables in OECD countries and we assess the consequences of exogenous shocks.

[21] Satisfying the existence condition $\gamma(1+\psi)(\delta - m_{ss}) > \psi$ requires $\delta - m_{ss} > 0$, which is intuitive: constant population with constant gross fertility requires a positive rate of population turnover.

We evaluate the transitional and the long-run effects of a permanent rise in the time cost of reproduction versus a permanent reduction in total immigration according to the specification $M(t) = \bar{M}$. While both these shocks may raise economic growth in the very long run, they also permanently reduce the mass of firms and the wage bill relative to assets, generating decades of slower growth and stagnating real interest rates. These effects yield net welfare losses for newborn generations entering the economy up to eighty years after the shock.

6.1 Baseline Parameters and Calibration

Our first objective is to determine a baseline parametrization whereby the theoretical model replicates the observed values of key macroeconomic variables. We focus on a hypothetical economy in steady state where the targeted variables assume the average values observed across OECD countries. Panel A in Table 1 considers a first list of six *endogenous* variables for which we calculate target values from available data (OECD, 2017) or empirical evidence: sources and identification methods are discussed in detail in Appendix E. Population size, $L_{ss} = 36,525,680$, matches the average population of OECD economies in 2015. The propensity to consume out of total wealth, $c/(a+h) = 0.03$, is the typical mid-range value of estimated long-run propensities for OECD countries and for the US. The consumption-assets ratio, $(C/A)_{ss} = 0.64$, matches the cross-country mean of the average ratio observed within each country (where such data exist) during the 1995-2015 period, calculated as households' final consumption divided by households' financial net worth. The mass of firms relative to population, $N_{ss}/L_{ss} = 0.0327$, equals the OECD-average number of firms in 2013 divided by the average population in the same year. The target growth rate of final output, determined by the asymptotic rate of wealth creation $(\dot{A}/A)_{ss} = 0.014$, is the implicit growth rate of real GDP according to the long-term forecasts published in OECD (2017). The target value of R&D propensity, 0.022, is given by the average ratio between R&D expenditures and gross domestic product observed in OECD countries during the 1995-2015 period.

Panel B in Table 1 lists our *preset parameters*, the values of which reflect available data or empirical estimates. The death probability $\delta = 0.016$ is the reciprocal of the average expected years of adult lifetime in OECD countries, $1/\delta = 62.5$. The long-run migration rate, $m_{ss} = 0.0023$, is the cross-country mean of the average net immigration rate observed within each economy during the 1973-2012 period. In the simulations, we match this target by setting the mass of immigrants $\bar{M} = 84,009$. The value of the elasticity of substitution across intermediates, $\epsilon = 4.3$, implies a mark-up for monopolistic firms equal to 1.3, in the middle of the range 1.2-1.4 suggested by international evidence.[22] The rates of time preference and product obsolescence, ρ and μ, are set so as to obtain plausible values of interest rates, private returns to households' assets, and profit discount rates for firms. The combination $\rho = 1.5\%$ and $\mu = 1\%$ generates the equilibrium interest rate $r_{ss} = 2.98\%$ and, hence, a fair-annuity rate $(r + \delta)$ on households' assets of 4.6%, and a profit-discount rate $(r+\mu)$ for firms of 4% in the steady state. The value of $\chi-1$ represents the elasticity of productivity to the mass of intermediate goods, a parameter for which Broda et al. (2006) provide structural estimates ranging from 0.05 to 0.20, where the lower bound applies to advanced economies and the upper bound to developing countries.[23] We adopt a conservative approach by setting initially the baseline value on the low end, $\chi - 1 = 0.05$, and then perform a sensitivity analysis with alternative values in the range 0.025-0.10. For parameter $\varkappa$, we set a baseline value of zero and then check the robustness of our results under alternative values.

Given the preset parameters, we calibrate the remaining six parameters listed in Panel C of Table 1 so as to match the six target values of the endogenous variables listed in Panel A. First, we consider the demographic block of the model: since the values δ and m_{ss} already determine steady state fertility b_{ss}, we identify (γ, ψ, η) by imposing the target values of L_{ss}, $(C/A)_{ss}$ and $c/(a+h)$ in the respective equations (36′), (39) and (10). Considering the supply-side, we identify $(\varphi, \omega, \theta)$ by imposing the target values of N_{ss}/L_{ss} and $(\dot{A}/A)_{ss}$ in equations (44) and (46), respectively, and by setting the target value of the R&D propensity, 2.2%, equal to the steady-state ratio between total wages paid to non-production

[22] The range 1.2-1.4 captures most estimates of *economy-wide* mark-ups for the US and UK economies – see Britton et al. (2000) and the further references cited in Appendix E. The empirical literature shows that sectoral mark-ups may obey wider ranges, with mean values typically higher in services and lower in manufacturing.

[23] There is growing empirical literature on trade estimating the 'gains from variety expansion' from highly disaggregated data on imported goods. Such estimates are highly model-specific (see Feenstra, 2010) and most econometric models identify such gains with a 'love-for-variety' parameter that appears in Dixit-Stiglitz preferences. Our theoretical model, instead, identifies the gains from variety with a technology parameter for which the closest empirical counterpart is the 'elasticity of productivity to variety' calculated by Broda et al. (2006).

workers, $w\left(\ell_N+\ell_Z\right)$, and aggregate income calculated from the budget constraint (7). This procedure yields the calibrated values of the parameters reported in Table 1, Panel C.

6.2 Steady State Results

The first row of panel D in Table 1 reports the steady-state values of the main variables of interest under the baseline parametrization. Besides the targeted values already discussed, we obtain the gross fertility rate $b_{ss}=1.37\%$ implied by the demographic law, and a ratio of total labor incomes to assets $(w\tilde{L}/A)_{ss}=0.62$ that is empirically plausible. The same panel considers six alternative parametrizations showing how the steady state changes in response to *ceteris paribus* variations in demographic parameters. The numerical results for higher mortality and stronger impatience confirm and extend our analytical findings (subsect. 5.3) on the opposite effects of δ and ρ: besides population and labor incomes relative to assets, also the mass of firms and the long-run growth rates move in opposite directions in response to the two shocks.

The third and fourth scenarios emphasize, instead, the similar consequences of reductions in $\bar{M}$ and increases in γ. Reduced immigration and increased reproduction costs produce a qualitatively different response of the fertility rate. The reduction in $\bar{M}$ increases b_{ss} because a permanent fall in net inflows is ultimately compensated by increased domestic births in the steady state. The increase in γ, instead, reduces transitional fertility rates via higher private costs of reproduction leaving the long-run rate b_{ss} unchanged. Despite the asymmetric fertility response, the two shocks bear qualitatively identical consequences on the other endogenous variables. Reduced immigration and increased reproduction costs reduce the long-run population level and drive down labor incomes relative to assets; the long-run mass of firms shrinks, and the reallocation of workers to vertical R&D boosts interest rates in the long run. Although both migration and child-cost shocks enhance wealth creation in the long run, the welfare consequences of such shocks are neither clear-cut nor symmetric across generations: we investigate this point in the next subsection by studying both the transitional and the long-run effects of large permanent shocks.

6.3 Demographic Shocks, Transition and Welfare

The calibrated model allows us to evaluate the transitional effects of exogenous shocks and their impact on intergenerational welfare. Differently from Table 1, this subsection considers large unexpected permanent shocks. We set year 2015 as our reference time zero and assume that the shocks hit the 'average OECD economy' from year 2020 onwards. We focus on the two scenarios of 'reduced immigration' and 'increased reproduction cost' bearing similar quantitative effects on steady-state population. The first scenario assumes a *migration shock* whereby the net inflows $\bar{M}$ fall permanently by 25% of the baseline value, from $84,009$ to $63,006$, which may be interpreted as an 'immigration barrier' set by a policymaker. The second scenario assumes a *child-cost shock* whereby γ permanently increases by 6% of its baseline value.

Panel A in Table 2 provides a summary comparison of the migration and child-cost shocks in terms of initial, short-to-medium-run and steady-state effects on selected variables. Figure 2 presents a detailed analysis of the transitional paths generated by the two shocks over a century-long horizon. As already noted, besides the different response of fertility rates, the two shocks produce qualitatively identical dynamics for all the variables of interest. The important new insight delivered by Table 2 and Figure 2 is that the impact of both shocks on growth-related variables in the short-to-medium run is *reversed* with respect to the steady state outcomes. Although reduced migration and increased reproduction costs raise wealth creation and interest rates in the very long run, the transition features several decades of slower growth and reduced rates of return. The reason is that the decline in population creates net exit of firms from the market during the whole transition, $\dot{N}/N < 0$, and the decline in the rate of horizontal innovations reduces wealth creation, $\dot{A}/A$, in the short-to-medium run. By the same mechanism, during the first part of the transition, interest rates decline and labor is reallocated from entry to production activities. The scope of these transitional effects is substantial because the convergence to the new steady state is slow. The bottom panel in Figure 2 shows that the negative demographic shocks substantially reduce *aggregate* consumption in the medium run: the smaller drop and the subsequent recovery that we observe in consumption *per capita* during this phase is actually due to the population decline rather than to faster output growth.

The existence of prolonged 'reversed growth effects' during the transition bears consequences for intergenerational welfare: the same shock may affect different cohorts in opposite ways. The first key question relates to the intensity and persistence of such reversed growth effects. In this respect, a crucial parameter is χ, which determines the relative contribution of the firms' net entry rate, $\dot{N}/N$, to the economy's rate of wealth creation,

$\dot{A}/A$. The transition paths in Figure 2 assume the benchmark value $\chi = 1.05$ and show that the reversed effect on wealth creation lasts 70 years: after the shock occurring in 2020, the growth rate of assets is below the baseline level until 2090. Assuming higher (lower) values of χ would further delay (anticipate) the switching date, i.e., the instant at which $\dot{A}/A$ crosses the pre-shock level from below. We investigate this point by setting alternative values for χ in a sensitivity analysis that addresses the second key question, namely, the welfare consequences of reversed growth effects.

Generations that happen to be alive when the shocks occur may experience net welfare losses despite the fact that such shocks are growth-enhancing in the long run. New cohorts entering the economy immediately after the shocks are particularly exposed because they heavily rely on labor incomes (newborn agents have zero financial wealth) and experience a productivity slowdown that reduces real wages for decades (with respect to the pre-shock baseline path). We tackle this aspect by calculating the value of a cohort-specific utility index,

$$EPW_j \equiv \int_j^{j+(1/\delta)} \left[\ln c_j\left(t\right) + \psi \ln b_j\left(t\right)\right] \cdot e^{-\rho(t-j)} dt, \tag{50}$$

which represents the *ex-post welfare* level enjoyed by a typical member of cohort j whose actual lifetime exactly coincides with life expectancy $1/\delta$. Table 2, panel B, reports the values of EPW_j for ten different cohorts born in the years $j = 2025, 2035, ..., 2115$, and compares their welfare levels in the three cases of interest: the 'no shock' scenario in which the economy remains in the baseline steady state forever, the migration shock, and the child-cost shock. We repeat this exercise under the alternative values $\chi = (1.025, 1.05, 1.10)$, where 1.05 is our benchmark, assuming the same initial stock of knowledge in all scenarios.[24] The general, robust conclusion delivered by Table 2 is that both shocks are welfare-reducing for a large set of newborn generations. All the cohorts born within a century after the child-cost shock suffer net welfare losses in the cases $\chi = (1.05, 1.10)$. Assuming $\chi = 1.025$, we observe net welfare gains for the cohorts born after 2100. The bottom-line is that although increased reproduction costs and reduced immigration may raise economic growth in the very long run, the transition to the new steady state features many decades of slower growth and lower lifetime utility for a long sequence of newborn generations.

[24] The numbers appearing in Table 2, panel B, are obtained by setting initial knowledge (at the time of the shock) equal to the value $Z\left(2020\right) = 478 \cdot 10^{-6}$ in all parametrizations; this value is chosen fro convenience as it yields a normalized level of consumption per capita $c\left(2015\right) = 1$ under $\chi = 1.025$.

7 Conclusion

The interaction between wealth dilution caused by disconnected generations and wealth creation driven by vertical and horizontal innovations delivers a theory of the population level that fully abstracts from Malthusian mechanisms. The nature of the economy's steady state, which features a constant endogenous population level and sustained endogenous output growth, suggests a novel, demography-based view of the long run equilibrium: demographic change originating in exogenous shocks that affect reproduction costs, life expectancy and migration, have a first-order impact on the functional distribution of income, productivity growth and intergenerational welfare. Our results challenge several predictions of traditional growth models and open the door to a wide set of quantitative applications to assess the macroeconomic impact of demographic phenomena and related public policies – e.g., population ageing and pension systems, child-cost subsidies and immigration policies – in a framework where output growth and the population level are closely interconnected even in the long run.

References

Aghion, P., Howitt, P. (1992). A Model of Growth Through Creative Destruction. *Econometrica* 60 (2): 323-351.

Barro, R., Becker, G. (1989). Fertility Choice in a Model of Economic Growth. *Econometrica* 57 (2): 481-501.

Blanchard, O. (1985). Debt, Deficits, and Finite Horizons. *Journal of Political Economy* 93: 223-247.

Bloom, D., Canning, D., Sevilla, J. (2003). *The Demographic Dividend*. RAND Corporation.

Brander, J., Taylor, M.S. (1998). The Simple Economics of Easter Island: A Ricardo-Malthus Model of Renewable Resource Use. *American Economic Review* 88 (1): 119-38.

Britton, E., Larsen, J., Small, I. (2000). Imperfect competition and the dynamics of markups. *Bank of England working papers* 110, Bank of England.

Broda, C., Greenfield, J., Weinstein, D. (2006). From Groundnuts to Globalization: A Structural Estimate of Trade and Growth. *NBER Working Papers* 12512, National Bureau of Economic Research.

Buiter, W. (1988). Death, Birth, Productivity Growth and Debt Neutrality. *Economic Journal* 98: 279-293.

Chu, A., Cozzi, G., Liao, C.-H. (2013). Endogenous fertility and human capital in a Schumpeterian growth model. *Journal of Population Economics* 26: 181-202.

Dinopoulos, E., Thompson, P. (1998). Schumpeterian Growth without Scale Effects. *Journal of Economic Growth* 3 (4): 313-335.

Eckstein, Z., Stern, S., Wolpin, K. (1988). Fertility choice, land, and the Malthusian hypothesis. *International Economic Review* 29: 353-361.

Ehrlich, I., Lui, F. (1997). The problem of population and growth: A review of the literature from Malthus to contemporary models of endogenous population and endogenous growth. *Journal of Economic Dynamics and Control* 21 (1): 205-242.

Faruqee, H. (2002). Population Ageing and its Macroeconomic Implications: A Framework for Analysis. *IMF Working Paper* 02/16.

Feenstra, R. (2010). Measuring the Gains from Trade under Monopolistic Competition. *Canadian Journal of Economics* 43 (1): 1-28.

Galor, O., Weil, D. (2000). Population, Technology, and Growth: From Malthusian Stagnation to the Demographic Transition and Beyond. *American Economic Review* 90 (4): 806-828.

Howitt, P. (1999). Steady Endogenous Growth with Population and R&D Inputs Growing. *Journal of Political Economy* 107 (4): 715-730.

Lucas, R. (2002). *Lectures on Economic Growth*. Cambridge, MA: Harvard University Press.

OECD (2017). *OECD Statistics*, www.stats.oecd.org (accessed July 2017). Organisation for Economic Co-operation and Development.

Peretto, P. (1998). Technological Change and Population Growth. *Journal of Economic Growth* 3: 283-311.

Peretto, P., Connolly, M. (2007). The manhattan metaphor. *Journal of Economic Growth* 12 (4): 329-350.

Peretto, P., Smulders, S. (2002). Technological distance growth and scale effects. *Economic Journal* 112: 603-624.

Peretto, P., Valente, S. (2015). Growth on a finite planet: resources, technology and population in the long run. *Journal of Economic Growth* 20: 305-331.

Romer, P. (1986). Increasing Returns and Long-run Growth. *Journal of Political Economy* 94 (5): 1002-1037.

Romer, P. (1990). Endogenous Technological Change. *Journal of Political Economy* 98 (5): 71-102.

Solow, R. (1956). A Contribution to the Theory of Economic Growth. *Quarterly Journal of Economics* 70 (1): 65-94.

Strulik, H., Weisdorf, J. (2008). Population, Food, and Knowledge: A Simple Unified Growth Theory. *Journal of Economic Growth* 13 (3): 195-216.

Weil, P. (1989). Overlapping families of infinitely-lived agents. *Journal of Public Economics* 38 (2): 183-198.

Wrigley, E.A. (1988). *Continuity, Chance and Change*. Cambridge: Cambridge University Press.

Yaari, M. (1965). Uncertain Lifetime, Life Insurance, and the Theory of the Consumer. *Review of Economic Studies* 32: 137-150.

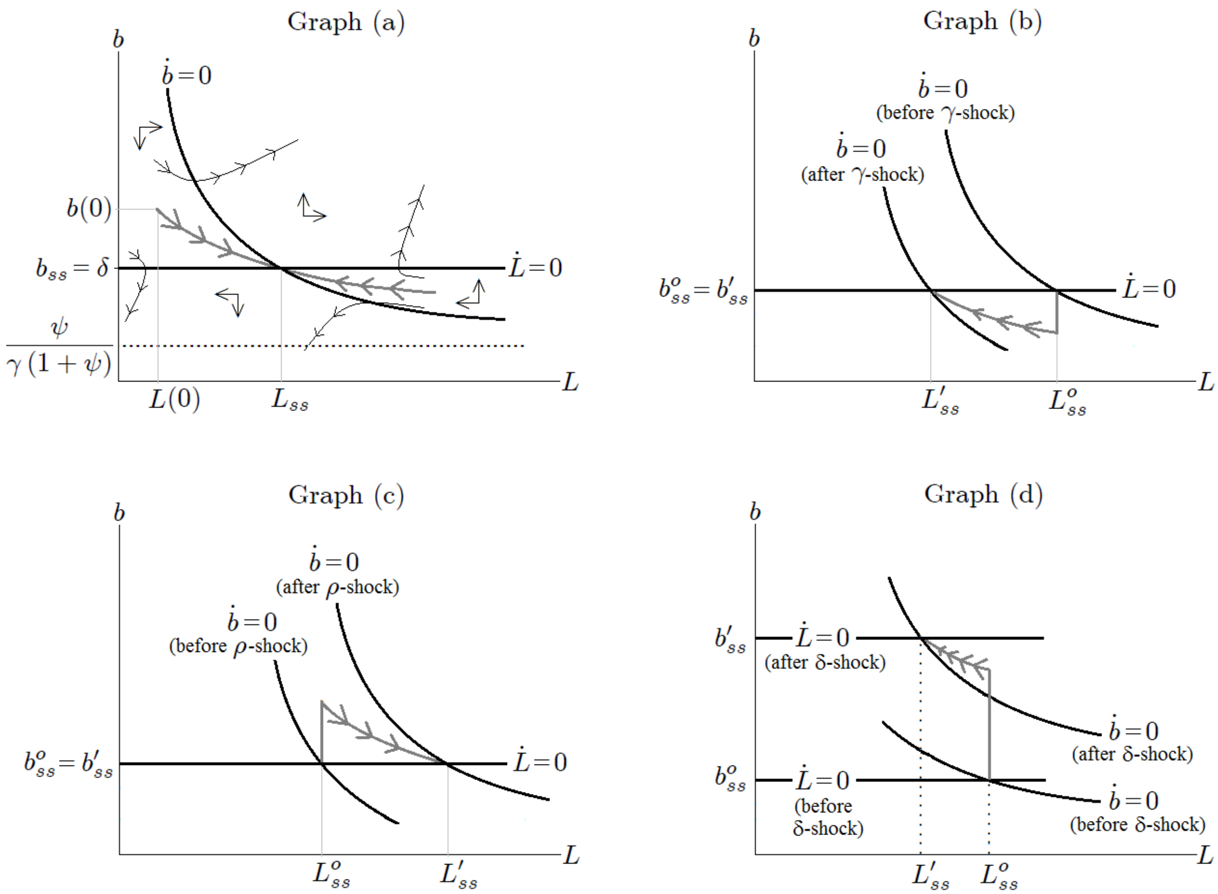

Figure 1. Graph (a): phase diagram of the dynamic system (29)-(30). Graph (b): Effects of an increase in γ. Graph (c): Effects of an increase in ρ. Graph (d): Effects of an increase in δ.

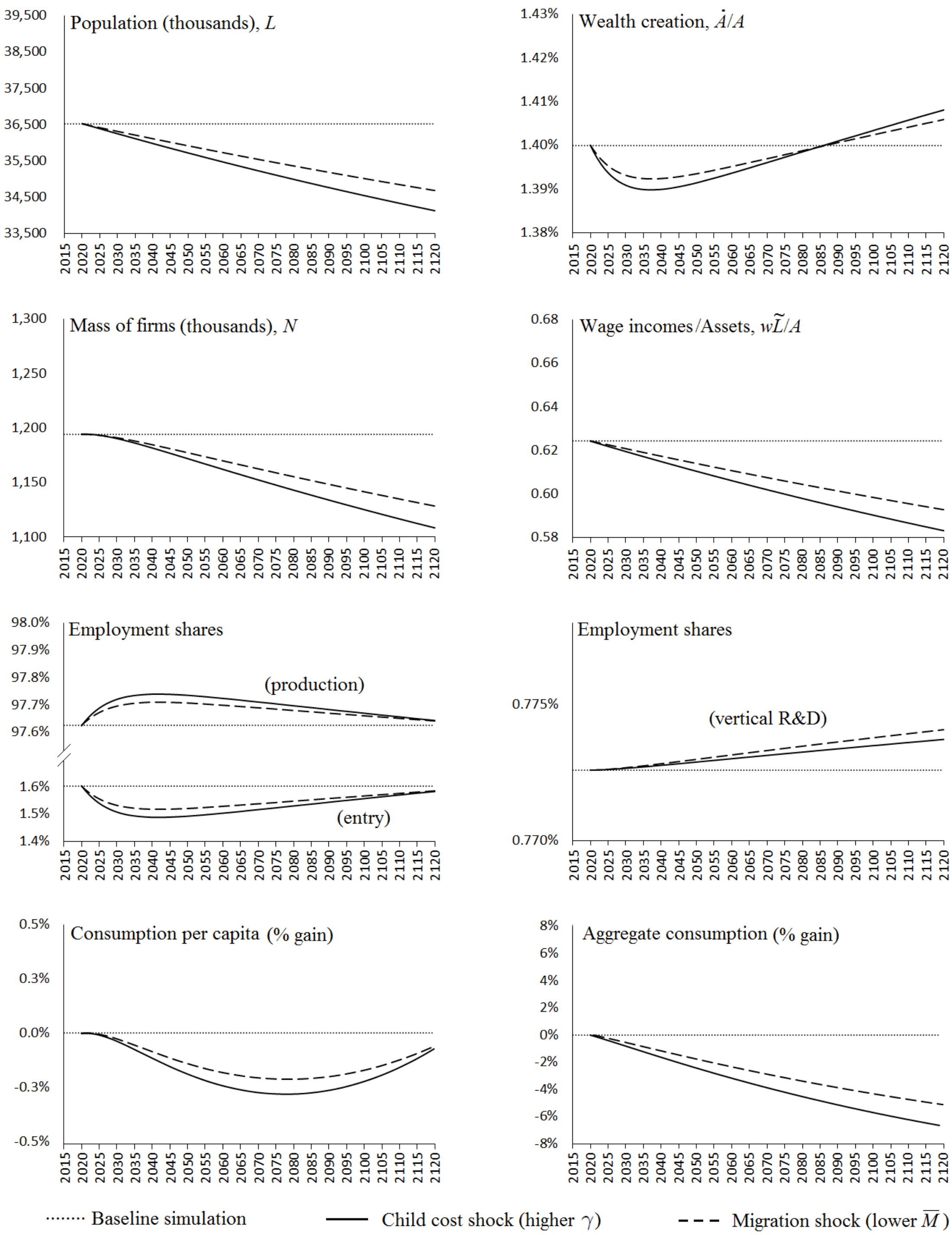

Figure 2. Transitional dynamics generated by exogenous increases in reproductions costs and by reduced immigration.

A. Targeted Variables		Population L_{ss}	Cons.Propensity $c/(a+h)$	Cons./Assets $(C/A)_{ss}$	Firms/Population N_{ss}/L_{ss}	Long-term Growth $(\dot{A}/A)_{ss}$	R&D Propensity $w(L_N+L_Z)/GDP$
Target values		36,525,680	0.03	0.64	0.0327	0.0140	0.022
(sources of target values)		(OECD data)	(OECD evidence)	(OECD data)	(OECD data)	(OECD forecast)	(OECD data)
Baseline simulation results		36,525,680	0.03	0.64	0.0327	0.0140	0.022
B. Preset parameters		δ	m_{ss}	ϵ	ρ	μ	χ
Baseline simulation parameters		0.016	0.0023	4.3	0.015	0.01	1.05
(identification)		(adult life exp.)	(migration rates)	(mark-up)	(utility disc.)	(profit disc.)	(variety gains)
C. Calibrated parameters		γ	ψ	η	φ	ω	θ
Baseline simulation parameters		2.412	0.033	$1.76 \cdot 10^{-8}$	5.602	6.127	0.01
(identification)		(cons./assets)	(cons.propens.)	(wage/assets)	(firms/population)	(long-term growth)	(R&D propens.)
D. Steady State Values		L_{ss}	b_{ss}	$(w\tilde{L}/A)_{ss}$	N_{ss}	$(\dot{A}/A)_{ss}$	r_{ss}
Baseline simulation		36,525,680	1.370%	0.624	1,194,390	1.400%	2.975%
Parameter variations:							
Higher mortality	(1% increase in δ)	34,443,479	1.372%	0.589	1,112,998	1.419%	3.000%
Stronger impatience	(1% increase in ρ)	36,569,637	1.370%	0.625	1,194,898	1.401%	2.992%
Reduced immigration	(1% decrease in $\bar{M}$)	36,207,651	1.370%	0.619	1,181,975	1.403%	2.978%
Higher child cost	(1% increase in γ)	34,739,921	1.358%	0.594	1,124,670	1.416%	2.995%

Table 1. Calibration of baseline parameters, steady state results and variations in parameters. See Appendix for details on sources and identification.

A. Exogenous shocks: transitional and long run effects

Shock	Migration shock (25% drop)					Child cost shock (6% increase)				
Parameter variation	$\overline{M}$ = 84,009 falls to $\overline{M}$ = 63,006 in year 2020					$\gamma = 2.41$ rises to $\gamma = 2.56$ in year 2020				
Endogenous variable	L (mln)	N (mln)	$w\tilde{L}/A$	$\dot{A}/A$	r	L (mln)	N (mln)	$w\tilde{L}/A$	$\dot{A}/A$	r
Year 2020 (pre-shock)	36.526	1.194	0.624	1.40%	2.98%	36.526	1.194	0.624	1.40%	2.98%
Year 2035	36.217	1.188	0.619	1.39%	2.97%	36.111	1.186	0.617	1.39%	2.96%
Year 2065	35.634	1.166	0.609	1.40%	2.97%	35.343	1.157	0.604	1.39%	2.97%
Year 2120	34.679	1.128	0.593	1.41%	2.98%	34.123	1.108	0.583	1.41%	2.99%
Steady state	28.587	0.884	0.488	1.49%	3.09%	28.199	0.869	0.482	1.50%	3.09%

B. Individual ex-post welfare

Cohort-specific index	EPW_{2025}	EPW_{2035}	EPW_{2045}	EPW_{2055}	EPW_{2065}	EPW_{2075}	EPW_{2085}	EPW_{2095}	EPW_{2105}	EPW_{2115}
No shock ($\chi = 1.05$)	27.91	33.59	39.26	44.94	50.62	56.30	61.98	67.66	73.33	79.01
Migration shock ($\chi = 1.05$)	27.83	33.49	39.15	44.82	50.50	56.19	61.88	67.58	73.28	79.00
Child cost shock ($\chi = 1.05$)	27.73	33.38	39.04	44.70	50.38	56.07	61.77	67.48	73.19	78.92
No shock ($\chi = 1.025$)	13.72	19.40	25.07	30.75	36.43	42.11	47.79	53.47	59.14	64.82
Migration shock ($\chi = 1.025$)	13.66	19.32	24.99	30.67	36.35	42.04	47.74	53.45	**59.16**	**64.88**
Child cost shock ($\chi = 1.025$)	13.56	19.22	24.89	30.56	36.25	41.95	47.65	53.37	59.09	**64.82**
No shock ($\chi = 1.10$)	56.28	61.96	67.64	73.32	79.00	84.68	90.35	96.03	101.7	107.4
Migration shock ($\chi = 1.10$)	56.17	61.82	67.47	73.13	78.80	84.47	90.15	95.84	101.5	107.2
Child cost shock ($\chi = 1.10$)	56.06	61.69	67.34	72.99	78.65	84.32	90.01	95.70	101.4	107.1

Table 2. Transitional, long-run and welfare effects of permanent shocks: reduced immigration versus increase in reproduction cost.

Appendix

General note. This Appendix collects all the proofs and derivations using the generalized model that includes migration (subsection 5.4). The model without migration, presented in sections 2-4 of the main text, is obtained as a special case of the generalized model by setting the immigration rate $m(t) = M(t)/L(t)$ equal to zero.

A Appendix: The Demographic Model

Household problem: derivation of (3) and (4). The current-value hamiltonian associated to the optimization problem solved by the j-th individual is

$$\log c_j(t) + \psi \log b_j(t) + \lambda^h(t)\left[(r(t)+\delta) a_j(t) + (1-\gamma b_j(t)) w(t) - p(t) c_j(t)\right],$$

where λ^h is the dynamic multiplier. The necessary conditions for maximization are

$$\lambda^h(t) p(t) c_j(t) = 1, \tag{A.1}$$

$$\lambda^h \gamma w(t) b_j(t) = \psi, \tag{A.2}$$

$$(\rho+\delta)\lambda^h(t) - \dot{\lambda}^h(t) = \lambda^h(t)(r+\delta), \tag{A.3}$$

$$\lim_{t\to\infty} \lambda^h(t) a_j(t) e^{-(\rho+\delta)(t-j)} = 0. \tag{A.4}$$

Combining (A.1) with (A.2), we obtain condition (4). Time-differentiating (A.1) and substituting the resulting expression in (A.3) yields (3). For future reference, also note that (A.3) and (A.4) imply the standard transversality condition

$$\lim_{t\to\infty} \frac{\dot{a}_j(t)}{a_j(t)} < \lim_{t\to\infty} r(t) + \delta, \tag{A.5}$$

according to which individual wealth should not grow faster than the effective interest rate.

Demographic law: derivation of (5). In the generalized model with migration, we define $L(t)$ as total domestic adult population at instant t, and use $B(t)$ to denote *domestic births*, i.e., the mass of children generated at instant t by the $L(t)$ adults that reside within the economy under study. Suppose for simplicity that migrants enter or leave the economy once-and-for-all at the beginning of their working age – which, in the present model, means that migration occurs immediately after birth. Denoting by $M(t)$ the net inflow of migrants in the economy, the total size at time t of the cohort that 'entered the economy' at time

j is denoted by $k_j(t)$. By construction, the size of cohort j at instant j equals the sum of domestic offspring and immigrants,

$$k_j(j) = B(j) + M(j). \tag{A.6}$$

Assuming that foreign immigrants exhibit the same death probability δ and the same preference for fertility as domestic residents, the size of cohort j at time t is then given by

$$k_j(t) = k_j(j) \cdot e^{-\delta(t-j)} = [B(j) + M(j)] \cdot e^{-\delta(t-j)}. \tag{A.7}$$

Given the definitions of total adult population and domestic births

$$L(t) \equiv \int_{-\infty}^{t} k_j(t)\, dj \quad \text{and} \quad B(t) \equiv \int_{-\infty}^{t} k_j(t)\, b_j(t)\, dj, \tag{A.8}$$

Using (A.6) to substitute $k_j(t)$ in the definition of $L(t)$, and differentiating the resulting expression with respect to t, we obtain the generalized demographic law

$$\dot{L}(t) = B(t) + M(t) - \delta L(t). \tag{A.9}$$

Setting net migration to zero, $M(t) = 0$, we have expression (5), i.e., the demographic law used in section 2. In the generalized model with migration, we define the *net immigration rate* $m(t) \equiv M(t)/L(t)$ and rewrite the demographic law (A.9) as

$$\frac{\dot{L}(t)}{L(t)} = b(t) + m(t) - \delta. \tag{A.10}$$

Aggregate wealth constraint: derivation of (7). Total assets and aggregate consumption are defined as

$$A(t) \equiv \int_{-\infty}^{t} k_j(t)\, a_j(t)\, dj \quad \text{and} \quad C(t) \equiv \int_{-\infty}^{t} k_j(t)\, c_j(t)\, dj. \tag{A.11}$$

From (A.11), the time-derivative of $A(t)$ equals

$$\dot{A}(t) = \frac{\mathrm{d}}{\mathrm{d}t} \int_{-\infty}^{t} k_j(t)\, a_j(t)\, dj = k_t(t)\, a_t(t) + \int_{-\infty}^{t} \left[k_j(t)\, \dot{a}_j(t) + \dot{k}_j(t)\, a_j(t)\right] dj,$$

where we can substitute $a_t(t) = 0$ (i.e., people entering the economy have zero assets and receive no bequests), as well as the transition law of cohort size $\dot{k}(j,t) = -\delta k(j,t)$ from (A.7), to obtain

$$\dot{A}(t) = \int_{-\infty}^{t} k_j(t)\, \dot{a}_j(t)\, dj - \delta A(t). \tag{A.12}$$

Using the budget constraint (2) to substitute $\dot{a}_j(t)$, and aggregating, expression (A.12) yields

$$\begin{aligned}\dot{A}(t) &= \int_{-\infty}^{t} k(j,t)\left[(r(t)+\delta)\cdot a(j,t)+(1-\gamma b(j,t))\cdot w(t)-p(t)c(j,t)\right]dj-\delta A(t)= \\ &= (r(t)+\delta)\cdot A(t)+w(t)L(t)-w(t)\gamma B(t)-p(t)C(t)-\delta A(t), \qquad \text{(A.13)}\end{aligned}$$

which, dividing both sides by $A(t)$, reduces to expression (7) in the main text.

Individual expenditure: derivation of (9). Integrating the individual budget constraint (2) forward in time, we obtain

$$a_j(t)=\int_t^{\infty}\left[p(s)c_j(s)-(1-\gamma b_j(s))w(s)\right]\cdot e^{-\int_t^s(r(v)+\delta)dv}ds.$$

Using (4) to substitute $\gamma b(j,s)w(s)$, and rearranging terms, the above expression yields

$$a_j(t)+\int_t^{\infty}w(s)\cdot e^{-\int_t^s(r(v)+\delta)dv}ds=(1+\psi)\int_t^{\infty}p(s)c_j(s)\cdot e^{-\int_t^s(r(v)+\delta)dv}ds. \qquad \text{(A.14)}$$

From the Keynes-Ramsey rule (3), we substitute $p(s)c_j(s)=p(t)c_j(t)\cdot e^{\int_t^s(r(v)-\rho)dv}$ in (A.14) and solve the residual integral, obtaining

$$a_j(t)+\int_t^{\infty}w(s)\cdot e^{-\int_t^s(r(v)+\delta)dv}ds=\frac{1+\psi}{\rho+\delta}\cdot p(t)c_j(t), \qquad \text{(A.15)}$$

which is expression (9) in the text.

Per capita expenditure: derivation of (10). Multiplying both sides of (A.15) by the size of cohort j, and integrating across cohorts, we obtain

$$\frac{1}{\rho+\delta}\cdot(1+\psi)p(t)\cdot\underbrace{\int_{-\infty}^{t}k_j(t)\cdot c_j(t)\,dj}_{C(t)}=\underbrace{\int_{-\infty}^{t}k_j(t)\cdot a_j(t)\,dj}_{A(t)}+\int_{-\infty}^{t}k_j(t)h(t)\,dj, \qquad \text{(A.16)}$$

where $C(t)$ is aggregate consumption as defined in (A.11). Since $h(t)$ is independent of j, we can rewrite (A.16) as

$$\frac{1}{\rho+\delta}(1+\psi)p(t)C(t)=A(t)+h(t)\underbrace{\int_{-\infty}^{t}k_j(t)\,dj}_{L(t)},$$

which, dividing both sides by $L(t)$, reduces to expression (7) in the main text.

Per capita consumption growth: derivation of (11). Substituting (4) to substitute $B(t)$ in (A.13), we have

$$\dot{A}(t)=r(t)A(t)+w(t)L(t)-(1+\psi)p(t)C(t). \qquad \text{(A.17)}$$

Equation (A.17) implies

$$\frac{A(t)}{L(t)} \cdot \left(\frac{\dot{A}(t)}{A(t)} - \frac{\dot{L}(t)}{L(t)} \right) = \frac{r(t)\, A(t)}{L(t)} + w(t) - (1+\psi) \frac{p(t)\, C(t)}{L(t)} - \frac{A(t)}{L(t)} \cdot \frac{\dot{L}(t)}{L(t)}. \quad \text{(A.18)}$$

From the definition of human wealth $h(t) = \int_t^\infty w(s) \cdot e^{-\int_t^s (r(v)+\delta) dv} ds$, we have the time derivative

$$\dot{h}(t) = (r(t) + \delta) \cdot h(t) - w(t). \quad \text{(A.19)}$$

Given these results, time-differentiate (10) to obtain

$$\frac{\mathrm{d}p(t)\, c(t)}{\mathrm{d}t} = \frac{\rho+\delta}{1+\psi} \left[\frac{A(t)}{L(t)} \left(\frac{\dot{A}(t)}{A(t)} - \frac{\dot{L}(t)}{L(t)} \right) + \dot{h}(t) \right],$$

where we can substitute (A.18) and (A.19), along with $\dot{L}(t) = B(t) + M(t) - \delta L(t)$ from (A.9), to get

$$\frac{\mathrm{d}p(t)\, c(t)}{\mathrm{d}t} = \frac{\rho+\delta}{1+\psi} \cdot \left[(r(t)+\delta) \left(\frac{A(t)}{L(t)} + h(t) \right) - (1+\psi) \frac{p(t)\, C(t)}{L(t)} - \frac{A(t)}{L(t)} \left(\frac{B(t)}{L(t)} + m(t) \right) \right]. \quad \text{(A.20)}$$

Substituting $B(t) = \frac{\psi}{\gamma} \cdot \frac{p(t) C(t)}{w(t)}$ from (6) in (A.20), and dividing both sides by $p_c c \equiv pC/L$ using (10), we have

$$\frac{\mathrm{d}p(t)\, c(t)}{\mathrm{d}t} \cdot \frac{1}{p(t)\, c(t)} = r(t) - \rho - \left[\frac{\rho+\delta}{1+\psi} \left(\frac{\psi}{\gamma} \frac{1}{w(t)} + \frac{m(t)}{p(t)\, c(t)} \right) \right] \frac{A(t)}{L(t)}. \quad \text{(A.21)}$$

Re-arranging the left hand side of (A.21) yields

$$\frac{\dot{c}(t)}{c(t)} + \frac{\dot{p}(t)}{p(t)} = r(t) - \rho - \frac{\rho+\delta}{1+\psi} \left(\frac{\psi}{\gamma} \frac{1}{w(t)} + \frac{m(t)}{p(t)\, c(t)} \right) a(t), \quad \text{(A.22)}$$

which holds in the generalized model with migration. Setting $m(t) = 0$ in (A.22) gives expression (11) in the main text.

Fertility dynamics: derivation of (14). Time-differentiation of (13) gives

$$\frac{\dot{b}(t)}{b(t)} = \frac{\dot{c}(t)}{c(t)} + \frac{\dot{p}(t)}{p(t)} - \frac{\dot{w}(t)}{w(t)}, \quad \text{(A.23)}$$

where we can substitute (A.22) to obtain

$$\frac{\dot{b}(t)}{b(t)} = r(t) - \rho - \frac{\rho+\delta}{1+\psi} \left(\frac{\psi}{\gamma} \frac{1}{w(t)} + \frac{m(t)}{p(t)\, c(t)} \right) a(t) - \frac{\dot{w}(t)}{w(t)}. \quad \text{(A.24)}$$

From the dynamic wealth constraint (7), the interest rate may be expressed as

$$r(t) = \frac{\dot{A}(t)}{A(t)} - \frac{w(t)\, L(t)}{A(t)} + \frac{w(t)\, \gamma B(t)}{A(t)} + \frac{p(t)\, C(t)}{A(t)},$$

or, equivalently,

$$r(t) = \frac{\dot{a}(t)}{a(t)} + \frac{\dot{L}(t)}{L(t)} - \frac{w(t)}{a(t)} + \gamma \frac{w(t)\, b(t)}{a(t)} + \frac{p(t)\, c(t)}{a(t)}. \tag{A.25}$$

Plugging (A.25) into (A.24) to eliminate $r(t)$, we have

$$\frac{\dot{b}}{b} = \frac{\dot{a}}{a} + \frac{\dot{L}}{L} - \frac{w}{a} + \gamma \frac{wb}{a} + \frac{pc}{a} - \rho - \frac{\rho+\delta}{1+\psi}\left(\frac{\psi}{\gamma}\frac{1}{w} + \frac{m}{pc}\right) a - \frac{\dot{w}}{w},$$

where we can use (A.10) to substitute $\dot{L}/L$, and substitute $pc = b\gamma w/\psi$ from (13), obtaining

$$\frac{\dot{b}}{b} = b\left(1 + \gamma \frac{1+\psi}{\psi}\frac{w}{a}\right) + m - \rho - \delta - \frac{\rho+\delta}{1+\psi}\left(\frac{\psi}{\gamma}\frac{1}{w} + \frac{m}{pc}\right) a - \frac{w}{a} + \frac{\dot{a}}{a} - \frac{\dot{w}}{w}, \tag{A.26}$$

which holds in the generalized model with migration. Setting $m = 0$ in (A.26) gives expression (14) in the main text.

Alternative Supply-Side Structures (subsection 2.4): (ii) Neoclassical models with physical capital. In neoclassical models, aggregate financial wealth coincides with the economy's capital stock K employed in production with labor under constant returns to scale. Assume for simplicity the Cobb-Douglas production function $Y = K^{\alpha}\left[(1-\gamma b)\, L\right]^{1-\alpha}$ where $(1-\gamma b)\, L$ is net labor supply. Profit maximization in the production sector imply that $a/w = (1-\gamma b)/(1-\alpha)$ in each instant, where $a = K/L$ is assets per capita. Substituting this result into the core fertility equation (14), we have

$$\frac{\dot{b}(t)}{b(t)} = b(t)\left(1 - \gamma b(t) + \gamma \frac{(1+\psi)(1-\alpha)}{\psi}\right) - (\rho+\delta)(1-\gamma b(t)) - (1-\alpha) - \frac{\psi(\rho+\delta)}{\gamma(1+\psi)(1-\alpha)} \cdot (1-\gamma b(t))^2$$

which is an autonomous equation in the fertility rate: denoting the right hand side by $\tilde{f}(b)$, the steady-state locus $\dot{b} = 0$ is represented by $\tilde{f}(b) = 0$ with

$$\begin{aligned} \frac{\partial \tilde{f}(b)}{\partial b} &> 0, \qquad \lim_{b\to+\infty} \tilde{f}(b) = +\infty, \\ \lim_{b\to 0} \tilde{f}(b) &= -(\rho+\delta+1-\alpha) - \frac{\psi(\rho+\delta)}{\gamma(1+\psi)(1-\alpha)} < 0, \end{aligned}$$

implying a unique unstable steady state b_{ss} that satisfies $\tilde{f}(b_{ss}) = 0$ and that is generally different from δ. We thus have the result that, combining the demographic structure of section 2 with the supply side of neoclassical models, the fertility rate jumps at the stationary value b_{ss} and is constant thereafter: the economy features endogenous constant population growth, which "pulls" constant aggregate capital growth in the long run, in line with the traditional notion of balanced growth. This result implies that combining our demographic structure with a neoclassical production structure does not produce a theory of the population level (the very special case in which b_{ss} coincides with δ may only happen by chance).

Alternative Supply-Side Structures (subsection 2.4): (iii) Constant returns to physical capital. Consider the following model with learning-by-doing as a modification of the neoclassical case discussed above. At the aggregate level, output is given by $Y = K^{\alpha}\left[T\cdot(1-\gamma b)\,L\right]^{1-\alpha}$ where T is labor productivity determined by an externality to be specified later, and wealth per capita is $a = K/L$. In this case, profit maximization with respect to employed labor units $(1-\gamma b)\,L$ yields $w = (1-\alpha)\,K^{\alpha}T^{1-\alpha}\,(1-\gamma b)^{-\alpha}\,L^{-\alpha}$ and, hence, the ratio

$$a/w = \frac{(1-\gamma b)^{\alpha}}{(1-\alpha)\,L^{1-\alpha}}\cdot\frac{K^{1-\alpha}}{T^{1-\alpha}} \quad \text{with} \quad T = \begin{cases} T_{(1)} \equiv K/\left[L\,(1-\gamma b)\right] \\ T_{(2)} \equiv K \end{cases}$$

where $T_{(1)}$ and $T_{(2)}$ are two alternative specifications of the learning-by-doing externality that capture the polar cases considered in the growth literature. Specification $T = T_{(2)}$ follows Romer's (1986) model in assuming that labor productivity grows with the aggregate capital stock, and implies the strong scale effect, namely, the prediction that population size increases permanently the economy's growth rate. Specification $T = T_{(1)}$ assumes, instead, that the externality is determined by the capital to labor ratio and is the easiest way to eliminate the strong scale effect from this class of models with constant returns to capital.[25] Considering $T = T_{(1)} = K/\left[L\,(1-\gamma b)\right]$, the wealth-wage ratio reduces to

$$a/w_{(1)} = \frac{1-\gamma b}{1-\alpha},$$

which is exactly the same result holding in the neoclassical models discussed above, implying that the fertility rate jumps at the stationary value b_{ss} and is constant thereafter. Therefore, the endogenous growth model with constant returns to capital under specification $T_{(1)}$ does not produce a theory of the population level (the very special case in which b_{ss} coincides with δ may only happen by chance). Next, consider $T = T_{(2)} = K$ as in Romer (1986). In this case, we have

$$a/w_{(2)} = \frac{(1-\gamma b)^{\alpha}}{(1-\alpha)\,L^{1-\alpha}}, \qquad \frac{\dot{a}}{a} - \frac{\dot{w}}{w} = -\alpha\frac{\gamma\dot{b}}{1-\gamma b} - (1-\alpha)\,\frac{\dot{L}}{L},$$

[25] A more general specification to obtain a scale-free model is $T \equiv K/\left[L\,(1-\gamma b)\right]^{\bar{\sigma}}$, which would not change our conclusions: we concentrate on the case $\bar{\sigma} = 1$ to preserve clarity.

which may be substituted in the core fertility equation (14) to obtain

$$\frac{\dot{b}}{b}=\frac{1}{1+\frac{\alpha b}{1-\gamma b}}\left[\alpha b+\frac{b\gamma\left(1+\psi\right)-\psi}{\psi}\cdot\frac{\left(1-\alpha\right)L^{1-\alpha}}{\left(1-\gamma b\right)^{\alpha}}-\rho-\alpha\delta-\frac{\psi\left(\rho+\delta\right)}{\gamma\left(1+\psi\right)}\cdot\frac{\left(1-\gamma b\right)^{\alpha}}{\left(1-\alpha\right)L^{1-\alpha}}\right]. \tag{A.27}$$

From (A.27), a steady state with constant fertility ($\dot{b}=0$) and constant population ($b=\delta$) requires satisfying

$$\delta+\frac{1-\alpha}{\left(1-\gamma\delta\right)^{\alpha}}\left[\frac{\gamma\delta\left(1+\psi\right)-\psi}{\psi}\right]\cdot L_{ss}^{1-\alpha}=\rho+\delta+\frac{\psi\left(\rho+\delta\right)}{\gamma\left(1+\psi\right)}\cdot\frac{\left(1-\gamma\delta\right)^{\alpha}}{\left(1-\alpha\right)}\cdot\frac{1}{L_{ss}^{1-\alpha}} \tag{A.28}$$

which shows that a positive steady-state population level $L_{ss}>0$ exists provided that $\gamma\delta\left(1+\psi\right)>\psi$. However, in such a steady state with constant population, the economy's growth rate[26]

$$\frac{\dot{Y}}{Y}=\frac{\dot{K}}{K}=\frac{Y}{K}-\frac{C}{K}=\left[\left(1-\gamma b_{ss}\right)L_{ss}\right]^{1-\alpha}-\frac{\gamma}{\psi}\frac{b_{ss}}{a/w_{(2)}} \tag{A.29}$$

can be rewritten as

$$\frac{\dot{Y}}{Y}=\frac{1-\gamma\delta-\frac{1-\alpha}{\psi}\gamma\delta}{\left(1-\gamma\delta\right)^{\alpha}}\cdot L_{ss}^{1-\alpha}. \tag{A.30}$$

Expression (A.27) delivers two main results. First, by combining the existence condition in (A.28) with the numerator in the right hand side of (A.30), we see that the existence of a steady state with constant population and constant output growth requires that parameters satisfy

$$\gamma\delta\left(1+\psi-\alpha\right)<\psi<\gamma\delta\left(1+\psi\right), \tag{A.31}$$

where the first inequality is the requirement for positive output growth and the second inequality is necessary for $L_{ss}>0$ to exist. Since the plausible range of values for α typically lies between 0.2 and 0.6, the set of parametrizations satisfying (A.31) is extremely narrow. The first conclusion is thus that if a steady-state with constant population under Romer's (1986) specification $T=T$, it is very unlikely to generate strictly positive output growth. The second conclusion is that the growth rate (A.30) is affected by the strong scale effect: in this model, any positive (negative) shock on the population level increases (decreases) permanently the economy's growth rate.

[26] The last term in (A.29) are given by $\frac{C}{K}=\frac{\gamma}{\psi}b\frac{w}{a}$, which follows from combining result (13) with $a=K/L$.

B Appendix: The Production Side

Profit maximization in the final sector. The profits of the final sector read

$$p\left(t\right)C\left(t\right)-\int_0^{N(t)} p_{xi}\left(t\right)x_i\left(t\right)di = N\left(t\right)^{\chi-\frac{\epsilon}{\epsilon-1}}\cdot\left(\int_0^{N(t)} x_i\left(t\right)^{\frac{\epsilon-1}{\epsilon}}di\right)^{\frac{\epsilon}{\epsilon-1}}-\int_0^{N(t)} p_{xi}\left(t\right)x_i\left(t\right)di,$$

and the first order conditions with respect to each quantity x_i read

$$p\left(t\right)N\left(t\right)^{\chi-\frac{\epsilon}{\epsilon-1}}\cdot\left(\int_0^{N(t)} x_i\left(t\right)^{\frac{\epsilon-1}{\epsilon}}di\right)^{\frac{\epsilon}{\epsilon-1}-1}\cdot x_i\left(t\right)^{-\frac{1}{\epsilon}}=p_{xi}\left(t\right). \tag{B.1}$$

Using technology (15), expression (B.1) may be rewritten as

$$p_{xi}\left(t\right)=\frac{p\left(t\right)C\left(t\right)}{\int_0^{N(t)} x_i\left(t\right)^{\frac{\epsilon-1}{\epsilon}}di}\cdot x_i\left(t\right)^{-\frac{1}{\epsilon}}=\breve{c}\left(t\right)x_i\left(t\right)^{-\frac{1}{\epsilon}} \tag{B.2}$$

where we have defined the term $\breve{c}\equiv pC/\int_0^N x_i^{\frac{\epsilon-1}{\epsilon}}di$.

Intermediate Producers: the incumbent's problem. The instantaneous profits of the typical monopolist read

$$\pi_{xi}\left(t\right)\equiv p_{xi}\left(t\right)x_i\left(t\right)-w\left(t\right)\ell_{xi}\left(t\right)-w\left(t\right)\ell_{zi}\left(t\right). \tag{B.3}$$

From the production function (17), labor employed in intermediate production by each incumbent firm equals

$$\ell_{xi}\left(t\right)=x_i\left(t\right)z_i\left(t\right)^{-\theta}+\varphi. \tag{B.4}$$

Using (B.4) and the demand schedule (B.2), we can rewrite the monopolist instantaneous profits (B.3) as

$$\pi_{xi}\left(t\right)=\breve{c}\left(t\right)x_i\left(t\right)^{\frac{\epsilon-1}{\epsilon}}-w\left(t\right)\left[x_i\left(t\right)z_i\left(t\right)^{-\theta}+\varphi\right]-w\left(t\right)\ell_{zi}\left(t\right), \tag{B.5}$$

where the term $\breve{c}\equiv pC/\int_0^N x_i^{\frac{\epsilon-1}{\epsilon}}di$ is taken as given by the i-th monopolist. Given the knowledge accumulation law (18), the incumbent solves the problem

$$\max_{\{x_i(t),\ell_{zi}(t)\}} V_i\left(t\right)=\int_t^\infty \pi_{xi}\left(v\right)e^{-\int_t^v(r(s)+\mu)ds}dv \text{ subject to } \dot{z}_i\left(t\right)=\omega Z\left(t\right)\cdot\ell_{zi}\left(t\right)$$

where public knowledge $Z\left(t\right)$ is also taken as given. The current-value hamiltonian at instant v associated with this problem is

$$\breve{c}\left(v\right)x_i\left(v\right)^{\frac{\epsilon-1}{\epsilon}}-w\left(v\right)\left[x_i\left(v\right)z_i\left(v\right)^{-\theta}+\varphi\right]-w\left(v\right)\ell_{zi}\left(v\right)+\lambda_i\left(v\right)\omega Z\left(v\right)\ell_{zi}\left(v\right),$$

where λ_i is the dynamic multiplier, x_i and ℓ_{zi} are control variables, and z_i is the state variable. The necessary conditions for maximization read

$$\begin{aligned}
\frac{\epsilon-1}{\epsilon}\check{c}(v)\,x_i(v)^{-\frac{1}{\epsilon}} &= w(v)\,z_i(v)^{-\theta}, &\text{(B.6)}\\
\lambda_i(v)\,\omega Z(v) &= w(v), &\text{(B.7)}\\
\frac{\dot{\lambda}_i(v)}{\lambda_i(v)} &= r(v)+\mu-\theta\frac{w(v)}{\lambda_i(v)}x_i(v)\,z_i(v)^{-\theta-1}. &\text{(B.8)}
\end{aligned}$$

In the left hand side of (B.6), we can substitute $\check{c}x_i^{-\frac{1}{\epsilon}}=p_{xi}$ from (B.2) to obtain the mark-up rule for the intermediate good's price at each instant t,

$$p_{xi}(t)=\frac{\epsilon}{\epsilon-1}w(t)\,z_i(t)^{-\theta}. \tag{B.9}$$

Multiplying both sides of (B.9) by x_i and substituting the technology (17), we obtain the profit-maximizing value of intermediate production

$$p_{xi}(t)\,x_i(t)=\frac{\epsilon}{\epsilon-1}\cdot w(t)\left(\ell_{xi}(t)-\varphi\right). \tag{B.10}$$

Time-differentating (B.7) and substituting the resulting expression in (B.8) to eliminate $\dot{\lambda}_i/\lambda_i$, we have

$$\frac{\dot{w}(t)}{w(t)}-\frac{\dot{Z}(t)}{Z(t)}=r(v)+\mu-\theta\omega Z(t)\,x_i(t)\,z_i(t)^{-\theta-1}=r(t)+\mu-\theta\omega Z(t)\frac{\ell_{xi}(t)-\varphi}{z_i(t)}, \tag{B.11}$$

where the central term makes use of (B.7) to eliminate λ_i, and the last term follows from (B.4). Using (B.10) to eliminate $\ell_{xi}-\varphi$ from (B.11), and rearranging terms, yields

$$r(t)+\mu=\theta\frac{\epsilon-1}{\epsilon}\frac{p_{xi}(t)\,x_i(t)}{z_i(t)}\omega\frac{Z(t)}{w(t)}+\frac{\dot{w}(t)}{w(t)}-\frac{\dot{Z}(t)}{Z(t)} \tag{B.12}$$

which is the firm's rate of return to vertical innovation shown as (21) in the main text.

Symmetric equilibrium. Our specification of public knowledge (19) implies that each firm faces equal initial conditions besides exploiting the same technology. Therefore, intermediate sector operates in a symmetric equilibrium where each firm $i\in[0,N(t)]$ exhibits

$$\begin{aligned}
z_i(t) &= Z(t), &\text{(B.13)}\\
x_i(t) &= Z(t)^{\theta}\cdot\left(\ell_{xi}(t)-\varphi\right), &\text{(B.14)}\\
p_{xi}(t) &= \frac{\epsilon}{\epsilon-1}w(t)\,Z(t)^{-\theta}, &\text{(B.15)}
\end{aligned}$$

at each point in time. From (15), symmetry implies that aggregate final output equals

$$C(t) = N(t)^{\chi - \frac{\epsilon}{\epsilon-1}} \cdot \left(\int_0^{N(t)} x_i(t)^{\frac{\epsilon-1}{\epsilon}} \, di \right)^{\frac{\epsilon}{\epsilon-1}} = N(t)^{\chi} x_i(t). \tag{B.16}$$

From the demand schedule (16), we have

$$x_i(t) = \frac{p(t)\, C(t)}{N(t)\, p_{xi}(t)}. \tag{B.17}$$

Combining (B.16) with (B.17), the price of intermediates is

$$p_{xi}(t) = p(t)\, N(t)^{\chi-1}. \tag{B.18}$$

Moreover, from (B.14) and (B.17), we have the aggregate value of final output as

$$p(t)\, C(t) = N(t)\, p_{xi}(t)\, Z(t)^{\theta} \cdot (\ell_{xi}(t) - \varphi),$$

where we can substitute p_{xi} using (B.14), obtaining

$$C(t) = N(t)^{\chi-1} \cdot Z(t)^{\theta} \cdot [N(t)\, \ell_{xi}(t) - N(t)\, \varphi]. \tag{B.19}$$

Labor market clearing: equation (24). Symmetry in the intermediate sector implies that aggregate employment in intermediate production and in vertical R&D equals

$$\int_0^{N(t)} (\ell_{xi}(t) + \ell_{zi}(t))\, di = N(t)\, \ell_{xi}(t) + N(t)\, \ell_{zi}(t).$$

Including employment in horizontal R&D, denoted by ℓ_N, and recalling that aggregate net labor supply equals $L - \gamma B$, the market-clearing condition for labor reads

$$N(t)\, \ell_{xi}(t) + N(t)\, \ell_{zi}(t) + \ell_N(t) = L(t) - \gamma B(t). \tag{B.20}$$

which, defining $\ell_X \equiv N\ell_{xi}$ and $\ell_Z \equiv N\ell_{zi}$, yields (24) in the main text.

Equilibrium wage: equation (25). From the profit-maximizing condition of intermediate producers (B.15) and the equilibrium price of intermediates (B.18), we have

$$p(t)\, N(t)^{\chi-1} = \frac{\epsilon}{\epsilon-1} w(t)\, Z(t)^{-\theta}, \tag{B.21}$$

which can be rearranged to obtain (25).

C Appendix: General Equilibrium

Dynamic system: derivation of (30). Slight manipulation allows us to rewrite (A.26) as

$$\frac{\dot{b}}{b}=\gamma\frac{1+\psi}{\psi}\cdot\frac{w}{a}b+(b+m-\delta)-\rho-\frac{\rho+\delta}{1+\psi}\left(\frac{\psi}{\gamma}+\frac{w}{c}m\right)\frac{a}{w}-\frac{w}{a}+\frac{\dot{a}}{a}-\frac{\dot{w}}{w},$$

where we can substitute $w/c=\psi/(\gamma b)$ from (13) and the growth rate of wage $\dot{w}/w=(\dot{a}/a)+(1-\varkappa)(\dot{L}/L)$ from (26), obtaining

$$\frac{\dot{b}}{b}=\gamma\frac{1+\psi}{\psi}\cdot\frac{w}{a}\cdot b+[b+m-\delta]-\rho-\frac{\psi}{\gamma}\frac{\rho+\delta}{1+\psi}\left(1+\frac{m}{b}\right)\frac{a}{w}-\frac{w}{a}-(1-\varkappa)\frac{\dot{L}}{L}. \tag{C.1}$$

Substituting the term in square brackets with $\dot{L}/L$ from (A.10), and substituting $w/a=\eta L^{1-\varkappa}$ from (26), equation (C.1) reduces to

$$\frac{\dot{b}}{b}=\left(\gamma\frac{1+\psi}{\psi}b-1\right)\eta L^{1-\varkappa}-\rho-\frac{\psi}{\gamma}\frac{\rho+\delta}{1+\psi}\left(1+\frac{m}{b}\right)\frac{1}{\eta L^{1-\varkappa}}+\varkappa(b+m-\delta), \tag{C.2}$$

which holds in the generalized model with migration. Setting $m=0$ in (C.2) gives expression (30) in the main text.

Dynamic system: derivation of (32) and proof of results (33). Setting $\dot{b}=0$ in (30) yields (32), which may equivalently be rewritten as

$$\bar{b}(L)=\delta\frac{\varkappa\delta+\eta L^{1-\varkappa}}{\varkappa\delta+\gamma\delta\frac{1+\psi}{\psi}\eta L^{1-\varkappa}}+\frac{\rho+\frac{\psi}{\gamma}\frac{\rho+\delta}{1+\psi}\frac{1}{\eta L^{1-\varkappa}}}{\varkappa+\gamma\frac{1+\psi}{\psi}\eta L^{1-\varkappa}}. \tag{C.3}$$

In the right hand side of (C.3), each term is negatively related to $L^{1-\varkappa}$ and this implies $\partial\bar{b}(L)/\partial L<0$. Taking the limits in (C.3) yields

$$\begin{aligned}\lim_{L\to 0}\bar{b}(L) &= \delta+\frac{\rho+\infty}{\varkappa}=+\infty,\\ \lim_{L\to+\infty}\bar{b}(L) &= \delta\frac{1}{\gamma\delta\frac{1+\psi}{\psi}}=\frac{\psi}{\gamma(1+\psi)},\end{aligned}$$

which complete the proof of results (33).

Proof of Proposition 1. This proof refers to the generalized model with migration under the specification of a constant migration rate, that is, $m(t)=\bar{m}$ exogenous and constant. The proof of Proposition 1 is therefore a special case with $\bar{m}=0$. Consider the generalized system given by equations (A.10) and (C.2) with $m(t)=\bar{m}$. The dynamic system is

$$\frac{\dot{L}}{L} = b+\bar{m}-\delta \tag{C.4}$$

$$\frac{\dot{b}}{b} = \left(\gamma\frac{1+\psi}{\psi}b-1\right)\eta L^{1-\varkappa}-\rho-\frac{\psi}{\gamma}\frac{\rho+\delta}{1+\psi}\frac{1+\frac{\bar{m}}{b}}{\eta L^{1-\varkappa}}+\varkappa(b+\bar{m}-\delta)\equiv F(b,L). \tag{C.5}$$

Imposing $\dot{L} = \dot{b} = 0$ in (C.4)-(C.5), we obtain

$$b_{ss} = \delta - \bar{m}, \tag{C.6}$$

$$\left(\gamma \frac{1+\psi}{\psi} b_{ss} - 1\right) \left(\eta L_{ss}^{1-\varkappa}\right)^2 = \rho \left(\eta L_{ss}^{1-\varkappa}\right) + \frac{\psi}{\gamma} \frac{\rho+\delta}{1+\psi} \frac{b_{ss}+\bar{m}}{b_{ss}}, \tag{C.7}$$

where the second-degree equation (C.7) has a unique positive root given by

$$\eta L_{ss}^{1-\varkappa} = \frac{\psi}{2} \frac{\rho + \sqrt{\rho^2 + 4\frac{\psi}{\gamma} \frac{\rho+\delta}{1+\psi} \frac{b_{ss}+\bar{m}}{b_{ss}} \left(\gamma \frac{1+\psi}{\psi} b_{ss} - 1\right)}}{\gamma (1+\psi) b_{ss} - \psi}. \tag{C.8}$$

Slightly re-arranging terms and substituting $b_{ss} = \delta - \bar{m}$ from (C.6) into (C.8) gives

$$L_{ss} = \left[\frac{\psi}{2\eta} \frac{\rho + \sqrt{\rho^2 + 4\delta(\rho+\delta)\left(1 - \frac{\psi}{\gamma(1+\psi)(\delta-\bar{m})}\right)}}{\gamma(1+\psi)(\delta-\bar{m}) - \psi} \right]^{\frac{1}{1-\varkappa}}. \tag{C.9}$$

Expressions (35)-(36) in Proposition 1 are obtained by setting $\bar{m} = 0$ in equations (C.6) and (C.9), respectively. Expression (37) follows from combining (C.9) with the equilibrium relationship $a/w = 1/\left(\eta L_{ss}^{1-\varkappa}\right)$ in (28). Concerning dynamic stability, the generalized system (C.4)-(C.5) exhibits

$$\left.\begin{pmatrix} \partial\dot{L}/\partial L & \partial\dot{L}/\partial b \\ \\ \partial\dot{b}/\partial L & \partial\dot{b}/\partial b \end{pmatrix}\right|_{(L,b)=(L_{ss},b_{ss})} = \begin{pmatrix} 0 & L_{ss} \\ \\ b_{ss} \cdot \left.\frac{\partial F}{\partial L}\right|_{ss} & \left.\frac{\partial F}{\partial b}\right|_{ss} \end{pmatrix}$$

with eigenvalues $(\lambda_{m,1}, \lambda_{m,2})$ given by

$$\lambda_m^2 - \lambda_m \frac{\partial F}{\partial b} - L_{ss} b_{ss} \frac{\partial F}{\partial L} = 0, \tag{C.10}$$

where both $\partial F(b,L)/\partial b$ and $\partial F(b,L)/\partial L$ are strictly positive in view of the definition of $F(b,L)$ in (C.5). Equation (C.10) thus exhibits two real roots $(\lambda_{m,1}, \lambda_{m,2})$ of opposite sign, which implies saddle-point stability of the steady state $(L,b) = (L_{ss}, b_{ss})$.

Predicted transitional dynamics and OECD data. The upper panel in Figure A1 reports two equivalent phase diagrams of the economy's core dynamics. Graph (a) is directly obtained from Figure 1 in the main text: considering the joint dynamics of fertility $b(t)$ and population $L(t)$, we can represent the stable arm of the saddle as a negative equilibrium relationship $b^E(L)$, with $\partial b^E(L)/\partial L < 0$. Graph (b) results from the following change of variables: dividing both sides of expression (38) by population L, we obtain

$$\frac{c(t)}{A(t)} = \frac{\gamma}{\psi} \cdot b(t) \eta L(t)^{-\varkappa} \text{ in each } t,$$

which allows us to rewrite the equilibrium c/A ratio as

$$c/A = \frac{\gamma}{\psi} \cdot b^E(L) \cdot L^{-\varkappa} \tag{C.11}$$

where $b^E(L)$ is the saddle path appearing in the benchmark phase diagram of graph (a). Since $\partial b^E(L)/\partial L < 0$ for any $\varkappa$, and $\partial L^{-\varkappa}/\partial L < 0$ for any $\varkappa > 0$, it follows from (C.11) that the model predicts $c(t)/A(t)$ to exhibit a positive relationship with the fertility rate and, overall, a negative co-movement with $L(t)$ regardless of whether $\varkappa$ is zero or strictly positive.

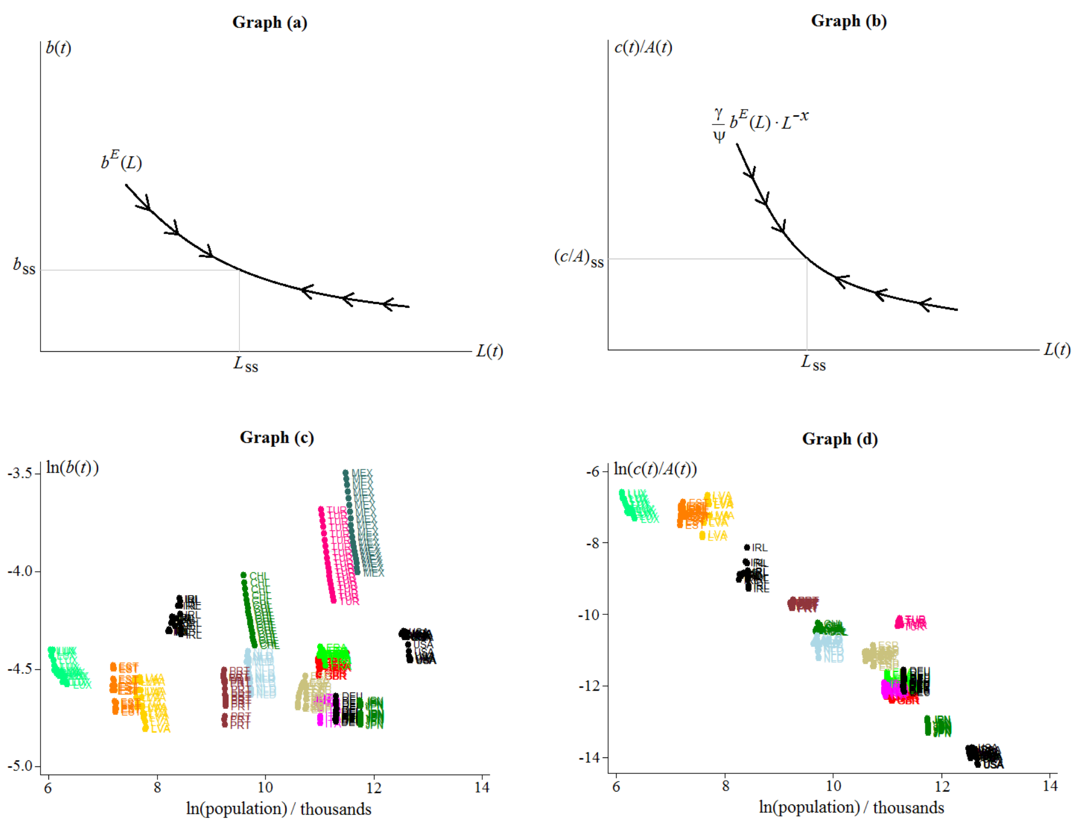

Figure A1. Graphs (a) and (b): transitional comovements predicted by the theoretical model. Graphs (c) and (d): scatter diagrams for selected OECD countries.

The lower panel of Figure A1 reports scatter diagrams in which the variables on the axes of graph (c) and graph (d) are the empirical counterparts (in logarithmic scale) of those on the axes of graph (a) and (b), respectively. We collected data for all OECD countries, interpreting b as the annual crude birth rate (i.e., number of births per person) reported by the World Bank (2017), L as total population reported by the UN (2015), c as Household

Final Consumption reported by OECD (2017a) divided by total population, and A with Financial Net Worth of Households reported by OECD (2017a). The longest time series for A available from OECD (2017a) cover the 1995-2016 period, with gaps for some countries and no observations for Mexico and New Zealand. The overall panel dataset thus covers the interval 1995-2016. For the sake of clarity, the scatter plots of graphs (c) and (d) in Figure A1 refer to a sub-sample of 16 countries, namely, Chile, Estonia, France, Germany, Great Britain, Ireland, Italy, Japan, Latvia, Luxembourg, Mexico, Portugal, Spain, The Netherlands, Turkey, United States. Graph (c) suggests a negative within-country co-movement between fertility rates and population size: this is confirmed by panel regressions with country fixed-effects computed for the *whole* OECD countries sample (see Table A.1, first and second columns). Graph (d) similarly suggests a strong pattern of negative co-movement between c/A and population size: this is also confirmed by fixed-effects panel regressions for the *whole* OECD countries sample (see Table A.1, third to fifth columns). Panel regression results, including country fixed effects and additionally controlling for country-specific time trends, are reported in Table A1.

Dependent variable Regressors	$\ln(b)$	$\ln(b)$	$\ln(c/A)$	$\ln(c/A)$	$\ln(c/A)$
$\ln(L)$	-0.503***	-0.421***	-1.554***	-0.512*	-0.401
	(0.0963)	(0.0831)	(0.439)	(0.258)	(0.313)
trend		-0.00107		-0.0110***	-0.00891*
		(0.00113)		(0.00379)	(0.00484)
$\ln(b)$					0.448*
					(0.253)
Constant	3.782**	2.444*	15.40**	-1.629	-1.443
	(1.572)	(1.354)	(7.199)	(4.230)	(4.394)
Observations	665	665	606	606	549
R-squared	0.118	0.121	0.115	0.177	0.148
Number of countries	35	35	33	33	33

Driscoll-Kraay standard errors in parentheses. The error structure is assumed to be heteroskedastic, autocorrelated and possibly correlated between countries. *** p<0.01, ** p<0.05, * p<0.1.

Table A.1: Fixed-effects panel regression results for OECD countries 1995-2015.

Innovation rates. The key relationships determining vertical and horizontal innovation rates are the rate of return to vertical innovation (B.12) and the entry technology (22) that we report here for simplicity,:

$$\frac{\dot{Z}(t)}{Z(t)} = \theta\frac{\epsilon-1}{\epsilon}\frac{p_{xi}(t)\,x_i(t)}{z_i(t)}\omega\frac{Z(t)}{w(t)}+\frac{\dot{w}(t)}{w(t)}-r(t)-\mu, \tag{C.12}$$

$$\frac{\dot{N}(t)}{N(t)} = \frac{\eta}{L(t)^{\varkappa}}\cdot\ell_N(t)-\mu. \tag{C.13}$$

Derivation of (41). First, by symmetry, simplify $Z/z_i = 1$ in (C.12) and rearrange terms to get

$$\frac{\dot{Z}(t)}{Z(t)} = \omega\theta\frac{\epsilon-1}{\epsilon}\frac{p_{xi}(t)\,x_i(t)}{w(t)} + \frac{\dot{w}(t)}{w(t)} - r(t) - \mu. \tag{C.14}$$

Next, by symmetry, substitute $p_{xi}x_i = \frac{pC}{N}$ from (B.17), and normalize $p(t) = 1$ to obtain

$$\frac{\dot{Z}(t)}{Z(t)} = \omega\theta\frac{\epsilon-1}{\epsilon}\frac{p(t)\,C(t)}{w(t)\,N(t)} + \frac{\dot{w}(t)}{w(t)} - r(t) - \mu = \omega\theta\frac{\epsilon-1}{\epsilon}\frac{c(t)}{w(t)}\cdot\frac{L(t)}{N(t)} + \frac{\dot{w}(t)}{w(t)} - r(t) - \mu. \tag{C.15}$$

Next, substitute $w = \eta A/L^{\varkappa}$ from (26) to obtain

$$\frac{\dot{Z}(t)}{Z(t)} = \frac{\omega\theta}{\eta}\cdot\frac{\epsilon-1}{\epsilon}\cdot\frac{c(t)}{a(t)}\cdot\frac{L(t)^{\varkappa}}{N(t)} + \frac{\dot{w}(t)}{w(t)} - r(t) - \mu. \tag{C.16}$$

The growth rate of wages follows from time-differentiating (26): plugging

$$\frac{\dot{w}(t)}{w(t)} = \frac{\dot{A}(t)}{A(t)} - \varkappa\frac{\dot{L}(t)}{L(t)} \tag{C.17}$$

into the right hand side of (C.16) gives

$$\frac{\dot{Z}(t)}{Z(t)} = \frac{\omega\theta}{\eta}\cdot\frac{\epsilon-1}{\epsilon}\cdot\frac{c(t)}{a(t)}\cdot\frac{L(t)^{\varkappa}}{N(t)} - \varkappa\frac{\dot{L}(t)}{L(t)} - \left[r(t) - \frac{\dot{A}(t)}{A(t)}\right] - \mu. \tag{C.18}$$

From the aggregate wealth constraint (7) it follows that the term in square brackets in (C.18) equals the excess of consumption expenditure over net labor income divided by financial assets,

$$r(t) - \frac{\dot{A}(t)}{A(t)} = \frac{c(t)}{a(t)} - (1-\gamma b(t))\cdot\frac{w(t)}{a(t)}. \tag{C.19}$$

Substituting (C.19) into (C.18), and rearranging terms, we have equation (41) in the main text.

Derivation of (42). Equation (42) follows from substituting the labor market clearing condition (24) in the entry technology (22). The derivation thus requires to consider equilibrium employment levels in each sector in turn. First, consider employment in intermediate production. From (B.19), we have that $\ell_X(t) = N(t)\,\ell_{xi}(t)$ may be written as

$$\ell_X(t) = \frac{C(t)}{Z(t)^{\theta}\,N(t)^{\chi-1}} + \varphi N(t),$$

where we can substitute $A\frac{\epsilon\eta L^{\varkappa}}{\epsilon-1} = Z^{\theta}N^{\chi-1}$ from (27) to obtain

$$\ell_X(t) = \underbrace{\frac{\epsilon-1}{\epsilon\eta L(t)^{\varkappa}}\cdot\frac{C(t)}{A(t)} + \varphi N(t)}_{\text{Employment in intermediate production}} \tag{C.20}$$

Second, consider employment in vertical R&D. From the vertical innovation technology (18), we have $\dot{Z}/Z = \omega \cdot \ell_Z/N$ and, hence,

$$\ell_Z(t) = \underbrace{\frac{1}{\omega} N(t) \frac{\dot{Z}(t)}{Z(t)}}_{\text{Employment in vertical R\&D}} \tag{C.21}$$

Third, using results (C.20) and (C.21), we can obtain employment in horizontal R&D residually from the market-clearing condition (24) as

$$\ell_N(t) = \underbrace{L(t) \cdot (1 - \gamma b(t))}_{\text{Net labor supply}} - \underbrace{\frac{\epsilon - 1}{\epsilon \eta L(t)^{\varkappa}} \cdot \frac{C(t)}{A(t)} - \varphi N(t)}_{\ell_X(t)} - \underbrace{\frac{1}{\omega} N(t) \frac{\dot{Z}(t)}{Z(t)}}_{\ell_Z(t)}. \tag{C.22}$$

Substituting (C.22) into the entry technology (22), we obtain the growth rate of the mass of firms (omitting time-arguments)

$$\frac{\dot{N}}{N} = \frac{\eta}{L^{\varkappa}} \ell_N - \mu = \frac{\eta}{L^{\varkappa}} \left[L \cdot (1 - \gamma b) - \frac{\epsilon - 1}{\epsilon \eta L^{\varkappa}} \cdot \frac{c}{a} - \varphi N - \frac{1}{\omega} N \frac{\dot{Z}}{Z} \right] - \mu. \tag{C.23}$$

Factorizing $\eta/L^{\varkappa}$ across the various terms in the square brackets, and recalling that $\eta L^{1-\varkappa} = w/a$ from (28), we obtain expression (42) in the main text.

Dynamics of the mass of firms: preliminaries. Before proceeding, we establish a useful result concerning the asymptotic behavior of the interest rate. From the wealth constraint (7), the gap between the interest rate and the growth rate of financial wealth can be written as in expression (C.19). In the steady state with constant population (L_{ss}, b_{ss}), all the terms appearing in the right hand side of (C.19) are constant (section 4.2). Since satisfying the transversality condition (A.5) requires that $r > \dot{A}/A$ hold in the long run, *satisfying the transversality condition (A.5) in the steady state* (L_{ss}, b_{ss}) *implies*

$$\lim_{t \to \infty} \left(r(t) - \frac{\dot{A}(t)}{A(t)} \right) = \left(\frac{c}{a} \right)_{ss} - (1 - \gamma b_{ss}) \left(\frac{w}{a} \right)_{ss} > 0. \tag{C.24}$$

We will exploit result (C.24) to characterize the long-run behavior of the mass of firms. The entry technology (22) implies that if horizontal R&D is non-operative, $\ell_N(t) = 0$, the mass of firms declines at the obsolescence rate μ and eventually reaches zero if non-operativeness continues to hold in the long run. Therefore, *a necessary condition to have a positive mass of firms in the long run,* $\lim_{t \to \infty} N(t) > 0$*, is that horizontal R&D is operative*, $\lim_{t \to \infty} \ell_N(t) > 0$*, and that employment in entry is succiently high to guarantee*

$$\lim_{t \to \infty} \frac{\dot{N}(t)}{N(t)} = \lim_{t \to \infty} \frac{\eta \ell_N(t)}{L(t)^{\varkappa}} - \mu \geqslant 0. \tag{C.25}$$

From (C.23), we can rewrite the growth rate of the mass of firms as (eliminating time arguments for simplicity)

$$\frac{\dot{N}}{N} = \left(1 - \frac{\epsilon - 1}{\epsilon L^{2\varkappa}}\right) \cdot \frac{c}{a} - \underbrace{\left[\frac{c}{a} - \frac{w}{a}(1 - \gamma b) + \mu\right]}_{r - \frac{\dot{A}}{A} + \mu} - \frac{\eta}{L^{\varkappa}} \left(\varphi + \frac{1}{\omega} \frac{\dot{Z}}{Z}\right) \cdot N \tag{C.26}$$

where the term in square brackets equals $r - \left(\dot{A}/A\right) + \mu$ from (C.19) and is asymptotically strictly positive from (C.24). Combining (C.25) with (C.26), it follows that *a necessary condition to have a positive mass of firms in the long run,* $\lim_{t\to\infty} N(t) > 0$*, is*

$$\left(1 - \frac{\epsilon - 1}{\epsilon L^{2\varkappa}}\right) \cdot \left(\frac{c}{a}\right)_{ss} - \left[\left(\frac{c}{a}\right)_{ss} - \left(\frac{w}{a}\right)_{ss}(1 - \gamma b_{ss}) + \mu\right] \geqslant \lim_{t\to\infty} \frac{\eta}{L^{\varkappa}} \left(\varphi + \frac{1}{\omega} \frac{\dot{Z}}{Z}\right) \cdot N, \tag{C.27}$$

where the left hand side is a finite constant and the right hand side is strictly positive because $L_{ss} > 0$, $\varphi > 0$, and $\dot{Z} \geqslant 0$. Expression (C.27) implies the following

Lemma 6 *Any long-run equilibrium with constant population* (L_{ss}, b_{ss}) *and a positive mass of firms* $\lim_{t\to\infty} N(t) > 0$ *exhibits a constant finite mass of firms* N_{ss}*:*

$$\lim_{t\to\infty} \frac{\dot{N}(t)}{N(t)} = 0, \quad \lim_{t\to\infty} N(t) = N_{ss} < \infty.$$

Proof. *The proof hinges on the fact that condition (C.27) can only be satisfied as a strict equality. The proof is by contradiction. Suppose that (C.27) holds as a strict inequality: from (C.26), the strict inequality would imply permanent growth in the mass of firms,* $\lim_{t\to\infty} \frac{\dot{N}(t)}{N(t)} > 0$*, and therefore* $\lim_{t\to\infty} N(t) = +\infty$*, which would make the right hand side of (C.27) grow to* $+\infty$*. This implies a contradiction of the hyopthesis of strict inequality because the left hand side of (C.27) is a finite constant. Having established that (C.27) can only hold as a strict equality, and being (C.27) necessary to have a positive mass of firms in the long run, it follows that any long-run equilibrium with constant population* (L_{ss}, b_{ss}) *and a positive mass of firms* $\lim_{t\to\infty} N(t) > 0$ *must satisfy*

$$\left(1 - \frac{\epsilon - 1}{\epsilon L_{ss}^{2\varkappa}}\right) \cdot \left(\frac{c}{a}\right)_{ss} - \left[\left(\frac{c}{a}\right)_{ss} - \left(\frac{w}{a}\right)_{ss}(1 - \gamma b_{ss}) + \mu\right] = \lim_{t\to\infty} \frac{\eta}{L^{\varkappa}} \left(\varphi + \frac{1}{\omega} \frac{\dot{Z}}{Z}\right) \cdot N, \tag{C.28}$$

which implies $\lim_{t\to\infty} \frac{\dot{N}(t)}{N(t)} = 0$ *from (C.26) and, hence, a constant finite mass of firms* $\lim_{t\to\infty} N(t) = N_{ss} < \infty$ *in the long run.*

Lemma 6 suggests two remarks. The first remark concerns the existence of the long-run equilibrium: since the right hand side of (C.28) is strictly positive, the existence of a long-run equilibrium with constant population (L_{ss}, b_{ss}) and with a positive mass of firms requires that the left hand side of (C.28) be strictly positive, that is,

$$\left(1-\frac{\epsilon-1}{\epsilon L_{ss}^{2\varkappa}}\right)\cdot\left(\frac{c}{a}\right)_{ss} > \underbrace{\left(\frac{c}{a}\right)_{ss}-\left(\frac{w}{a}\right)_{ss}(1-\gamma b_{ss})+\mu}_{\lim_{t\to\infty} r(t)-\frac{\dot{A}(t)}{A(t)}+\mu}, \tag{C.29}$$

where the right hand side is strictly positive by (C.24). The second remark is that (C.26) and Lemma 6 determine a general expression for the steady-state mass of firms

$$\lim_{t\to\infty} N(t) = N_{ss} = \frac{\left(1-\frac{\epsilon-1}{\epsilon L_{ss}^{2\varkappa}}\right)\cdot\left(\frac{c}{a}\right)_{ss}-\left[\left(\frac{c}{a}\right)_{ss}-\left(\frac{w}{a}\right)_{ss}(1-\gamma b_{ss})+\mu\right]}{\frac{\eta}{L_{ss}^{\varkappa}}\left(\varphi+\frac{1}{\omega}g_Z^{ss}\right)} > 0, \tag{C.30}$$

where $g_Z^{ss} \equiv \lim_{t\to\infty}\left(\dot{Z}(t)/Z(t)\right)$ is the asymptotic rate of vertical innovations: just like Lemma 6, expression (C.30) holds regardless of whether vertical R&D is operative, $g_Z^{ss} > 0$, or non-operative, $g_Z^{ss} = 0$, in the long run. In order to complete the analysis of the interactions between horizontal and vertical innovations, it is necessary to consider the two cases separately. We begin by discussing the case with non-operative vertical R&D, $g_Z^{ss} = 0$.

Dynamics of the mass of firms: non-operative vertical R&D. Suppose that vertical R&D is non-operative in the long run, i.e., there exists a finite instant t_0 such that $\dot{Z}(t)/Z(t) = 0$ in each $t \in [t_0, \infty)$. In this case, expression (C.26) implies that, as long as horizontal R&D is operative, $\ell_N(t) > 0$, the mass of firms obeys the logistic process with time-varying coefficients

$$\frac{\dot{N}(t)}{N(t)} = \underbrace{\left(1-\frac{\epsilon-1}{\epsilon L(t)^{2\varkappa}}\right)\cdot\frac{c(t)}{a(t)}-\left[\frac{c(t)}{a(t)}-\frac{w(t)}{a(t)}(1-\gamma b(t))+\mu\right]}_{\bar{q}_1(t)}-\underbrace{\frac{\varphi\eta}{L(t)^{\varkappa}}}_{\bar{q}_2(t)}\cdot N(t) \tag{C.31}$$

where both $\bar{q}_1(t)$ and $\bar{q}_2(t) > 0$ approach finite steady-state values, $\bar{q}_1^{ss}$ and $\bar{q}_2^{ss} > 0$, in the steady state with constant population (L_{ss}, b_{ss}). If the necessary condition for the existence of a steady state with positive mass of firms (C.29) is satisfied, we have $\bar{q}_1^{ss} > \bar{q}_2^{ss} > 0$, which implies a stable steady state for the mass of firms $\lim_{t\to\infty} N(t) = N_{ss} = \bar{q}_1^{ss}/\bar{q}_2^{ss}$,given by expression (C.30) with $g_Z^{ss} = 0$.

Dynamics of the mass of firms with operative vertical R&D / proof of Proposition 2). Suppose that vertical R&D is operative in the long run, i.e., there exists a finite instant t_0 such that $\dot{Z}(t)/Z(t) > 0$ in each $t \in [t_0, \infty)$. Assuming that the steady state with constant population (L_{ss}, b_{ss}) and positive mass of firms $N_{ss} > 0$, the equilibrium rate of vertical innovation (41) converges to the positive constant

$$\lim_{t\to\infty} \frac{\dot{Z}(t)}{Z(t)} = g_Z^{ss} \equiv (1-\gamma b_{ss})\left(\frac{w}{a}\right)_{ss} + \left[\left(\frac{\epsilon-1}{\epsilon}\right)\frac{\omega\theta}{\eta}\frac{L_{ss}^{\varkappa}}{N_{ss}} - 1\right]\left(\frac{c}{a}\right)_{ss} - \mu > 0. \quad \text{(C.32)}$$

The asymptotic condition for the operativess of vertical R&D is thus given by

$$\frac{\epsilon-1}{\epsilon}\frac{\omega\theta}{\eta}\frac{L_{ss}^{\varkappa}}{N_{ss}}\left(\frac{c}{a}\right)_{ss} > \underbrace{\left(\frac{c}{a}\right)_{ss} - \left(\frac{w}{a}\right)_{ss}(1-\gamma b_{ss}) + \mu}_{\lim_{t\to\infty} r(t) - \frac{\dot{A}(t)}{A(t)} + \mu}, \quad \text{(C.33)}$$

where the right hand side is strictly positive by (C.24). Assuming that parameters are such that the conditions for the asymptotic operativeness of horizontal and vertical R&D, (C.29) and (C.33), are both satisfied, the mass of firms converges to the steady state level N_{ss} given by (C.30) with $\lim_{t\to\infty} \frac{\dot{Z}(t)}{Z(t)} = g_Z^{ss} > 0$. During the transitional interval $t \in [t_1, \infty)$ in which $\ell_N(t) > 0$ and $\dot{Z}(t)/Z(t) > 0$, the mass of firms obeys the logistic process with time-varying coefficients

$$\frac{\dot{N}}{N} = \underbrace{\left(1 - \frac{\epsilon-1}{\epsilon L^{2\varkappa}}\right)\cdot\frac{c}{a} - \left[\frac{c}{a} - \frac{w}{a}(1-\gamma b) + \mu\right]}_{q_1(t)} - \underbrace{\frac{\eta}{L^{\varkappa}}\left(\varphi + \frac{1}{\omega}\frac{\dot{Z}}{Z}\right)}_{q_2(t)} \cdot N \quad \text{(C.34)}$$

where both $q_1(t)$ and $q_2(t) > 0$ approach finite steady-state values, q_1^{ss} and $q_2^{ss} > 0$, in the steady state with constant population (L_{ss}, b_{ss}). If the necessary condition for the existence of a steady state with positive mass of firms (C.29) is satisfied, we have $q_1^{ss} > q_2^{ss} > 0$ which implies a stable steady state for the mass of firms

$$\lim_{t\to\infty} N(t) = N_{ss} = q_1^{ss}/q_2^{ss}, \quad \text{(C.35)}$$

given by expression (C.30). We can re-express (C.34) and (C.35) in terms of the underlying parameters as follows. From (C.34), the asymptotic values of $q_1(t)$ and $q_2(t)$ are given by

$$q_1^{ss} = \lim_{t\to\infty} q_1(t) = \left(\frac{w}{a}\right)_{ss}(1-\gamma b) - \frac{\epsilon-1}{\epsilon L_{ss}^{2\varkappa}}\cdot\left(\frac{c}{a}\right)_{ss} - \mu, \quad \text{(C.36)}$$

$$q_2^{ss} = \lim_{t\to\infty} q_2(t) = \frac{\eta}{L_{ss}^{\varkappa}}\left(\varphi + \frac{1}{\omega}g_Z^{ss}\right). \quad \text{(C.37)}$$

In equation (C.36), we substitute $w/a = \eta L^{1-\varkappa}$ from (28) and $c/a = \frac{\gamma}{\psi} b\eta L^{1-\varkappa}$ from (38) to obtain

$$q_1^{ss} = \lim_{t\to\infty} q_1(t) = (1-\gamma b_{ss})\,\eta L_{ss}^{1-\varkappa} - \frac{\epsilon-1}{\epsilon}\cdot\frac{\gamma}{\psi} b_{ss}\eta L_{ss}^{1-3\varkappa} - \mu. \quad \text{(C.38)}$$

In equation (C.37), we substitute g_Z^{ss} from (C.32) and, again, $w/a = \eta L^{1-\varkappa}$ from (28) and $c/a = \frac{\gamma}{\psi} b\eta L^{1-\varkappa}$ from (38), to obtain

$$q_2^{ss} = \lim_{t\to\infty} q_2(t) = \frac{\eta}{L_{ss}^{\varkappa}} \left\{ \varphi + \frac{1}{\omega} \left[\left(1 - \frac{1+\psi}{\psi} \gamma b_{ss} \right) \eta L_{ss}^{1-\varkappa} + \frac{\epsilon - 1}{\epsilon} \frac{\omega\theta}{\psi} \gamma b_{ss} \frac{L_{ss}}{N_{ss}} - \mu \right] \right\}. \tag{C.39}$$

Setting the steady state $\dot{N} = 0$ in (C.34), and substituting (C.38) and (C.39) in the resulting condition $q_1^{ss} = q_2^{ss} N_{ss}$, we obtain the steady-state mass of firms

$$N_{ss} = \frac{\eta L_{ss}^{1-\varkappa} \left[1 - \gamma b_{ss} - \left(\frac{1}{L_{ss}^{2\varkappa}} + \theta \right) \cdot \frac{\epsilon - 1}{\epsilon} \frac{\gamma}{\psi} b_{ss} \right] - \mu}{\varphi - \frac{1}{\omega} \left[\frac{(1+\psi)\gamma b_{ss} - \psi}{\psi} \eta L_{ss}^{1-\varkappa} + \mu \right]} \cdot \frac{L_{ss}^{\varkappa}}{\eta}. \tag{C.40}$$

From (C.29) and (C.33), the same parameter restrictions guaranteeing operativeness in both R&D activities imply that the denominator and the numerator in (C.40) are both strictly positive. Since the numerator is increasing in L_{ss} and the denominator is decreasing in L_{ss} for any $\varkappa \in [0, 1)$, we have $\mathrm{d}N_{ss}/\mathrm{d}L_{ss} > 0$ for any $\varkappa \in [0, 1)$.

D Appendix: Demographic Shocks, Income Shares and Migration

Proof of Proposition 3. The comparative-statics results with respect to b_{ss} are self-evident: from $b_{ss} = \delta$ we have $\mathrm{d}b_{ss}/\mathrm{d}\gamma = 0$, $\mathrm{d}b_{ss}/\mathrm{d}\rho = 0$, and $\mathrm{d}b_{ss}/\mathrm{d}\delta > 0$. The comparative-statics results for L_{ss} can be proved by defining the parameter

$$\Omega \equiv \gamma (1 + \psi) \delta, \tag{D.1}$$

and by rewriting the steady-state population level given by (36) as

$$L_{ss} \equiv \left(\frac{\psi}{\eta 2} \right)^{\frac{1}{1-\varkappa}} \left[\frac{\rho + \sqrt{\rho^2 + 4\delta(\rho + \delta) \frac{\Omega - \psi}{\Omega}}}{\Omega - \psi} \right]^{\frac{1}{1-\varkappa}}. \tag{D.2}$$

Little manipulation of the term in square brackets yields

$$L_{ss} \equiv \left(\frac{\psi}{\eta 2} \right)^{\frac{1}{1-\varkappa}} \left[\frac{\rho}{\Omega - \psi} + \sqrt{\left(\frac{\rho}{\Omega - \psi} \right)^2 + \frac{4\delta(\rho + \delta)}{\Omega(\Omega - \psi)}} \right]^{\frac{1}{1-\varkappa}}. \tag{D.3}$$

Expression (D.3) implies $\partial L_{ss}/\partial\Omega < 0$ and thereby

$$\frac{\mathrm{d}L_{ss}}{\mathrm{d}\gamma} = \frac{\partial L_{ss}}{\partial\Omega} \cdot \frac{\partial\Omega}{\partial\gamma} = (1 + \psi)\delta \cdot \frac{\partial L_{ss}}{\partial\Omega} < 0.$$

Since Ω is independent of ρ, expression (D.3) directly implies $\mathrm{d}L_{ss}/\mathrm{d}\rho > 0$. Concerning variations in δ, we have $\partial\Omega/\partial\delta = \gamma(1+\psi) > 0$ but the overall sign of $\mathrm{d}L_{ss}/\mathrm{d}\delta$ depends on the last term inside the square root in (D.3), which is apparently ambiguous: total differentiation of that last term shows that

$$\operatorname{sign}\frac{\partial}{\partial\delta}\left[\frac{4\delta(\rho+\delta)}{\Omega(\Omega-\psi)}\right] = \operatorname{sign}\left\{\frac{\rho+2\delta}{\delta(\rho+\delta)} - \frac{\gamma(1+\psi)}{\Omega}\cdot\frac{2\Omega-\psi}{\Omega-\psi}\right\}. \tag{D.4}$$

The term in curly brackets in (D.4) can be shown to be strictly negative by contradiction: if it were positive, we would have

$$\frac{\rho+2\delta}{\delta(\rho+\delta)} \geqslant \frac{\gamma(1+\psi)}{\Omega}\cdot\frac{2\Omega-\psi}{\Omega-\psi}, \tag{D.5}$$

which, after substituting $\Omega \equiv \gamma(1+\psi)\delta$ and rearranging terms, would yield

$$\begin{aligned}
\frac{\rho+2\delta}{\rho+\delta} &\geqslant \frac{2\gamma(1+\psi)\delta-\psi}{\gamma(1+\psi)\delta-\psi},\\
1+\frac{\delta}{\rho+\delta} &\geqslant 1+\frac{\gamma(1+\psi)\delta}{\gamma(1+\psi)\delta-\psi},\\
\frac{\delta}{\rho+\delta} &\geqslant \frac{\gamma(1+\psi)\delta}{\gamma(1+\psi)\delta-\psi}.
\end{aligned}$$

The last inequality cannot be true since the left hand side is strictly less than unity whereas the right hand strictly exceeds unity. This contradiction implies that (D.5) is always violated and, hence, the term in curly brackets in (D.4) is strictly negative; this in turn implies that the term $\frac{4\delta(\rho+\delta)}{\Omega(\Omega-\psi)}$ in expression (D.3) – like all the other ratios appearing inside the square brackets – declines with δ, so that $\mathrm{d}L_{ss}/\mathrm{d}\delta < 0$.

Proof of Proposition 4. The steady-state values of the three ratios considered in the Proposition may be written as

$$\lim_{t\to\infty}\frac{w(t)L(t)(1-\gamma b(t))}{A(t)} = \lim_{t\to\infty}\frac{w(t)}{a(t)}(1-\gamma b(t)) = \frac{1}{(a/w)_{ss}}\cdot(1-\gamma b_{ss}), \tag{D.6}$$

$$\lim_{t\to\infty}\frac{C(t)}{A(t)} = \lim_{t\to\infty}\frac{c(t)}{a(t)} = \left(\frac{c}{a}\right)_{ss} = \frac{\gamma}{\psi}\cdot\frac{b_{ss}}{(a/w)_{ss}}, \tag{D.7}$$

$$\lim_{t\to\infty}\frac{w(t)L(t)(1-\gamma b(t))}{C(t)} = \lim_{t\to\infty}\frac{\frac{w(t)L(t)(1-\gamma b(t))}{A(t)}}{\frac{C(t)}{A(t)}} = \psi\cdot\frac{1-\gamma b_{ss}}{\gamma b_{ss}}, \tag{D.8}$$

where (D.7) follows from (39) and (D.8) follows from the ratio between (D.6) and (D.7). Assuming zero migration, we can substitute in the three expressions above the steady-state fertility rate $b_{ss} = \delta$ from (35), which proves Proposition 4.

Derivation of (47). From (36) and (37), the wage-wealth ratio wL/A in the steady state equals $\lim_{t\to\infty} w(t)L(t)/A(t) = 1/(a/w)_{ss} = \eta L_{ss}^{1-\varkappa}$ so that the signs of $\frac{\mathrm{d}}{\mathrm{d}\rho}\lim_{t\to\infty}\frac{w(t)L(t)}{A(t)}$ and $\frac{\mathrm{d}}{\mathrm{d}\delta}\lim_{t\to\infty}\frac{w(t)L(t)}{A(t)}$ are respectively given by $\frac{\mathrm{d}}{\mathrm{d}\rho}L_{ss} > 0$ and $\frac{\mathrm{d}}{\mathrm{d}\delta}L_{ss} < 0$, as proved in Proposition 3.

Derivation of (48). Blanchard's (1985) model with neoclassical technology is analyzed in depth in advanced textbooks, e.g., Barro and Sala-i-Martin (2004: Ch. 3). Consumption per worker and assets per worker obey the dynamic system reported in Barro and Sala-i-Martin (2004), page 183, equations [3.89]-[3.90], which can be written using our current notation as

$$\begin{aligned}\dot{a} &= f(a) - c - na,\\ \frac{\dot{c}}{c} &= f'(a) - \rho - (\delta+n)(\rho+\delta)\frac{a}{c},\end{aligned}$$

where a is capital per worker, $f(a)$ is output per worker, and n is the population growth rate. The simultaneous steady state is defined by the condition

$$\left(\frac{f(a_{ss})}{a_{ss}} - n\right)\left(f'(a_{ss}) - \rho\right) = (\delta+n)(\rho+\delta), \tag{D.9}$$

where, given positive but diminishing marginal returns to capital (that is, $f'(a) > 0$ and $f''(a) < 0$), the left hand side is strictly decreasing in a. Therefore, exogenous increases in ρ and δ that increase the right hand side of (D.9) imply a lower value of steady-state capital per worker a_{ss},

$$\frac{\mathrm{d}}{\mathrm{d}\rho}a_{ss} < 0 \text{ and } \frac{\mathrm{d}}{\mathrm{d}\delta}a_{ss} < 0. \tag{D.10}$$

Since the wage rate is given by the (standard neoclassical model's) marginal condition $w = f(a) - af'(a)$, we have

$$\lim_{t\to\infty}\frac{w(t)L(t)}{A(t)} = \frac{f(a) - af'(a)}{a} = \frac{f(a_{ss})}{a_{ss}} - f'(a_{ss}),$$

where the last term is strictly decreasing in a_{ss} and thus implies $\frac{\mathrm{d}}{\mathrm{d}\rho}\lim_{t\to\infty}\frac{w(t)L(t)}{A(t)} > 0$ and $\frac{\mathrm{d}}{\mathrm{d}\delta}\lim_{t\to\infty}\frac{w(t)L(t)}{A(t)} > 0$ in Blanchard's (1985) model.

Migration: list of modifications. Introducing migration requires modifying three sets of expressions. First, in section 2, we need to modify (5), (11), (12), and (14). Second, in section 4, we need to modify the expressions (29)-(34) and Proposition 1 accordingly. Third, the expressions in Proposition 4 also require minor changes. We describe each sections's modifications in turn below.

Migration: modifications to section 2. First, the demographic law (5) must be replaced by (5′) as derived above in equation (A.9). Second, equation (11) must be replaced by

$$\frac{\dot{c}(t)}{c(t)}+\frac{\dot{p}(t)}{p(t)}=r(t)-\rho-\frac{\psi(\rho+\delta)}{\gamma(1+\psi)}\frac{b(t)+m(t)}{b(t)}\frac{a(t)}{w(t)}, \tag{D.11}$$

which is obtained by substituting $pc=b\gamma w/\psi$ from (13) into the generalized equation (A.22). Third, expression (14) must be replaced by the generalized equation (A.26). Fourth, the augmented rate of financial wealth dilution appearing in (12′) is simply obtained from the last term in (D.11) above, which equals the non-augmented rate of financial dilution (12) multiplied by $\frac{b+m}{b}$.

Migration: modifications to section 4. The dynamic system (29)-(30) must be replaced by the generalized system given by (A.10) and (C.2), which we report here for simplicity:

$$\frac{\dot{L}}{L} = b+m-\delta \tag{D.12}$$

$$\frac{\dot{b}}{b} = \left(\gamma\frac{1+\psi}{\psi}b-1\right)\eta L^{1-\varkappa}-\rho-\frac{\psi}{\gamma}\frac{\rho+\delta}{1+\psi}\left(1+\frac{m}{b}\right)\frac{1}{\eta L^{1-\varkappa}}+\varkappa(b+m-\delta) \tag{D.13}$$

where m is potentially time-varying. Note that, from (D.12)-(D.13), any simultaneous steady state $\dot{L}=\dot{b}=\dot{m}=0$ characterized by (L_{ss},b_{ss},m_{ss}) must obey the general stationarity conditions

$$b_{ss} = \delta-m_{ss}, \tag{D.14}$$

$$L_{ss} = \left[\frac{\psi}{\eta 2}\cdot\frac{\rho+\sqrt{\rho^2+4\delta(\rho+\delta)\left(1-\frac{\psi}{\gamma(1+\psi)(\delta-m_{ss})}\right)}}{\gamma(1+\psi)(\delta-m_{ss})-\psi}\right]^{\frac{1}{1-\varkappa}}, \tag{D.15}$$

where the denominator in expression (D.15) implies that a general existence condition for the steady state is

$$\gamma(1+\psi)(\delta-m_{ss})>\psi. \tag{D.16}$$

The remainder of the dynamic analysis depends on how the migration flows are specified. The next two Lemmas consider the cases of constant migration rate, $m(t)=\bar{m}$, and constant migration flows $M(t)=\bar{M}$, respectively.

Lemma 7 *(Steady state with constant migration rate) Assume $m(t)=\bar{m}$ in each instant t. Provided that $\gamma(1+\psi)(\delta-\bar{m})>\psi$, the simultaneous steady state (L_{ss},b_{ss}) of system*

(D.12)-(D.13) exists, is unique, and is saddle-point stable. ***Proof***: *With a constant migration rate $m(t) = \bar{m}$, system (D.12)-(D.13) is autonomous in $(L(t), b(t))$. The simultaneous steady-state (L_{ss}, b_{ss}), provided it exists, is saddle-point stable because the proof of Proposition 1 already includes a constant migration rate. What remains to be proven is existence and uniqueness of the steady state. The general existence condition (D.16) becomes*

$$\gamma(1+\psi)(\delta - \bar{m}) > \psi. \tag{D.17}$$

Setting $\dot{L} = 0$ in (D.12) and $\dot{b} = 0$ in (D.13) we obtain the respective stationary loci

$$b = \delta - \bar{m}, \tag{D.18}$$

$$b^2 = \frac{\bar{m} \cdot \frac{\psi}{\gamma} \frac{\rho+\delta}{1+\psi}}{\frac{\gamma(1+\psi)}{\psi}\left(\eta L^{1-\varkappa}\right)^2 + \varkappa \eta L^{1-\varkappa}} + b \cdot \left\{ \frac{\varkappa(\delta - \bar{m}) + \eta L^{1-\varkappa}}{\varkappa + \frac{\gamma(1+\psi)}{\psi}\eta L^{1-\varkappa}} + \frac{\rho + \frac{\psi}{\gamma}\frac{\rho+\delta}{1+\psi}}{\frac{\gamma(1+\psi)}{\psi}\left(\eta L^{1-\varkappa}\right)^2 + \varkappa \eta L^{1-\varkappa}} \right\} \tag{D.19}$$

Expressions (D.18)-(D.19) are very similar to the stationary loci (31)-(32) of the model without migration and indeed exhibit the same general properties, as shown in Figure A2, diagram (a). The locus $\dot{L} = 0$, given by (D.18), is a straight horizontal line in the (L, b) plane. The locus $\dot{b} = 0$, given by (D.19), satisfies the same properties (33) that hold in the model without migration. To prove the last statement, note that the right hand side of (D.19) can be defined as a function $\Xi_2(b; L)$, which is linear in b with positive intercept and positive slope. Therefore, for each $L > 0$, there exists a unique value of $b > 0$ satisfying $\dot{b} = 0$ that is determined by the fixed point $\bar{b}(L) \equiv \arg\text{solve}\left\{b^2 = \Xi_2(b; L)\right\}$. The shape of the locus is determined by letting L vary and finding the corresponding fixed point $\bar{b}(L)$. In this respect, imposing that (D.17) holds, we can show that $\Xi_2(b; L)$ is negatively related to L in both intercept and slope terms – in particular, the slope coefficient of $\Xi_2(b; L)$, represented by the term in curly brackets in (D.19), satisfies

$$\frac{\partial}{\partial\left(\eta L^{1-\varkappa}\right)} \cdot \frac{\varkappa(\delta - \bar{m}) + \eta L^{1-\varkappa}}{\varkappa + \frac{\gamma(1+\psi)}{\psi}\eta L^{1-\varkappa}} = \frac{\varkappa \cdot \left[1 - (\delta - \bar{m})\frac{\gamma(1+\psi)}{\psi}\right]}{\left[\varkappa + \frac{\gamma(1+\psi)}{\psi}\eta L^{1-\varkappa}\right]^2} < 0.$$

Therefore, the fixed point $\bar{b}(L) \equiv \arg\text{solve}\left\{b^2 = \Xi_2(b; L)\right\}$ is such that $\frac{d}{dL}\bar{b}(L) < 0$. That is, the $\dot{b} = 0$ locus is strictly declining. As regards the horizontal asymptote of the $\dot{b} = 0$ locus, note that

$$\lim_{L \to +\infty} \Xi_2(b; L) = \frac{\varkappa(\delta - \bar{m}) + \eta L^{1-\varkappa}}{\varkappa + \frac{\gamma(1+\psi)}{\psi}\eta L^{1-\varkappa}} b = \frac{\psi}{\gamma(1+\psi)} b$$

where the last term follows from l'Hospital rule. This implies that the fixed point takes the limiting value

$$\lim_{L\to\infty} \bar{b}\left(L\right) = \arg\mathrm{solve}\left\{b^2 = \frac{\psi}{\gamma\left(1+\psi\right)}b\right\} = \frac{\psi}{\gamma\left(1+\psi\right)}.$$

The above results imply that $\gamma\left(1+\psi\right)\left(\delta-\bar{m}\right)>\psi$ guarantees existence and uniqueness of the steady state (L_{ss}, b_{ss}). The stable arm representing the joint equilibrium path of (L, b) is described in Figure A2, diagram (a).

Lemma 8 *(Steady state with constant migration level) Assume $M\left(t\right)=\bar{M}$ in each instant t. Provided that $\gamma\left(1+\psi\right)\delta>\psi$ the simultaneous steady state (L_{ss}, b_{ss}) exists, it is saddle-point stable, and it is associated to a steady-state migration rate $m_{ss}=\bar{M}/L_{ss}$ satisfying $\gamma\left(1+\psi\right)\left(\delta-m_{ss}\right)>\psi$.* ***Proof:*** *With a constant migration level $M\left(t\right)=\bar{M}$, system (D.12)-(D.13) becomes*

$$\frac{\dot{L}}{L} = b+\frac{\bar{M}}{L}-\delta \tag{D.20}$$

$$\frac{\dot{b}}{b} = \left(\gamma\frac{1+\psi}{\psi}b-1\right)\eta L^{1-\varkappa}-\rho-\frac{\psi}{\gamma}\frac{\rho+\delta}{1+\psi}\left(1+\frac{\bar{M}}{bL}\right)\frac{1}{\eta L^{1-\varkappa}}+\varkappa\left(b+\frac{\bar{M}}{L}-\delta\right) \tag{D.21}$$

and exhibits the stationary loci

$$b = \delta-\frac{\bar{M}}{L}, \tag{D.22}$$

$$b^2 = \frac{1}{L}\frac{\bar{M}\cdot\frac{\psi}{\gamma}\frac{\rho+\delta}{1+\psi}}{\frac{\gamma(1+\psi)}{\psi}\left(\eta L^{1-\varkappa}\right)^2+\varkappa\eta L^{1-\varkappa}}+b\cdot\left\{\frac{\varkappa\left(\delta-\frac{\bar{M}}{L}\right)+\eta L^{1-\varkappa}}{\varkappa+\frac{\gamma(1+\psi)}{\psi}\eta L^{1-\varkappa}}+\frac{\rho+\frac{\psi}{\gamma}\frac{\rho+\delta}{1+\psi}}{\frac{\gamma(1+\psi)}{\psi}\left(\eta L^{1-\varkappa}\right)^2+\varkappa\eta L^{1-\varkappa}}\right\} \tag{D.23}$$

The relevant difference with the previous case of constant migration rate can be immediately verified from Figure A2, diagram (b): the $\dot{L}=0$ locus given by (D.22) is not a horizontal straight line but rather an increasing concave curve in the (L, b) plane with horizontal asymptote

$$\lim_{L\to\infty} b_{(\dot{b}=0)} = \delta, \tag{D.24}$$

and exists in the positive orthant provided that

$$L \geqslant \bar{M}/\delta. \tag{D.25}$$

Inequality (D.25) is an obvious existence condition for the steady state: it requires that the migration rate $m=\bar{M}/L$ does not exceed the mortality rate δ and is infact an implicit *requirement for the general condition (D.17) to hold. As regards the* $\dot{b} = 0$ *locus, the analysis is similar to the case with constant migration rate: the right hand side of (D.23) can be denoted by* $\Xi_3(b;L)$*, that is, a linear function of* b *with positive intercept and slope. Therefore, for each* $L > 0$*, there is a unique fixed point* $\bar{b}(L) \equiv \arg\mathrm{solve}\left\{b^2 = \Xi_3(b;L)\right\}$ *determining a unique point* $\left(L, \bar{b}(L)\right)$ *of the* $\dot{b} = 0$ *locus in the relevant phase plane. The slope coefficient of* $\Xi_3(b;L)$*, represented by the term in curly brackets in (D.23), is decreasing in* L *and satisfies*

$$\lim_{L\to+\infty}\left\{\frac{\varkappa\left(\delta-\frac{\bar{M}}{L}\right)+\eta L^{1-\varkappa}}{\varkappa+\frac{\gamma(1+\psi)}{\psi}\eta L^{1-\varkappa}}+\frac{\rho+\frac{\psi}{\gamma}\frac{\rho+\delta}{1+\psi}}{\frac{\gamma(1+\psi)}{\psi}\left(\eta L^{1-\varkappa}\right)^2+\varkappa\eta L^{1-\varkappa}}\right\} = \frac{\varkappa\bar{M}\lim_{L\to+\infty}\frac{1}{L^{2-\varkappa}}+(1-\varkappa)\eta}{\frac{\gamma(1+\psi)}{\psi}(1-\varkappa)\eta} = \frac{\psi}{\gamma(1+\psi)}$$

which implies, again,

$$\lim_{L\to+\infty}\Xi_3(b;L) = \frac{\psi}{\gamma(1+\psi)}b \tag{D.26}$$

From (D.26), the $\dot{b} = 0$ *locus has a horizontal asymptote given by the limiting value of the fixed point*

$$\lim_{L\to\infty}\bar{b}(L) = \arg\mathrm{solve}\left\{b^2 = \frac{\psi}{\gamma(1+\psi)}b\right\} = \frac{\psi}{\gamma(1+\psi)}, \tag{D.27}$$

which is the same property found in the other cases without migration and with a constant migration rate. Combining (D.27) with (D.24), we obtain the phase diagram reported in Figure A2, diagram (b): the existence and uniqueness condition for a simultaneous steady state (L_{ss}, b_{ss}) *with constant migration level* $\bar{M}$ *is the same as in the model without migration (cf. inequality (34) in the main text), that is,* $\gamma(1+\psi)\delta > \psi$*. The additional requirement is that the steady state is actually feasible iff the population level* L *is large enough to satisfy* $L \geqslant \bar{M}/\delta$ *from (D.25). Note that when the economy approaches the steady state* (L_{ss}, b_{ss})*, also the migration rate endogenously determined by* $m(t) = \bar{M}/L(t)$*, approaches its own steady state* $m_{ss} \equiv \bar{M}/L_{ss}$*. As regards the stability of the steady state, following the same steps as in the proof of Proposition 1, system (D.20)-(D.21) exhibits* $\partial\dot{L}/\partial L$*,* $\partial\dot{L}/\partial b$*,* $\partial\dot{b}/\partial L$ *and* $\partial\dot{b}/\partial b$ *all strictly positive and has real eigenvalues displaying opposite signs, implying that* (L_{ss}, b_{ss}) *is a saddle-point. The stable arm representing the joint equilibrium path of* (L, b) *is described in Figure A2, diagram (b).*

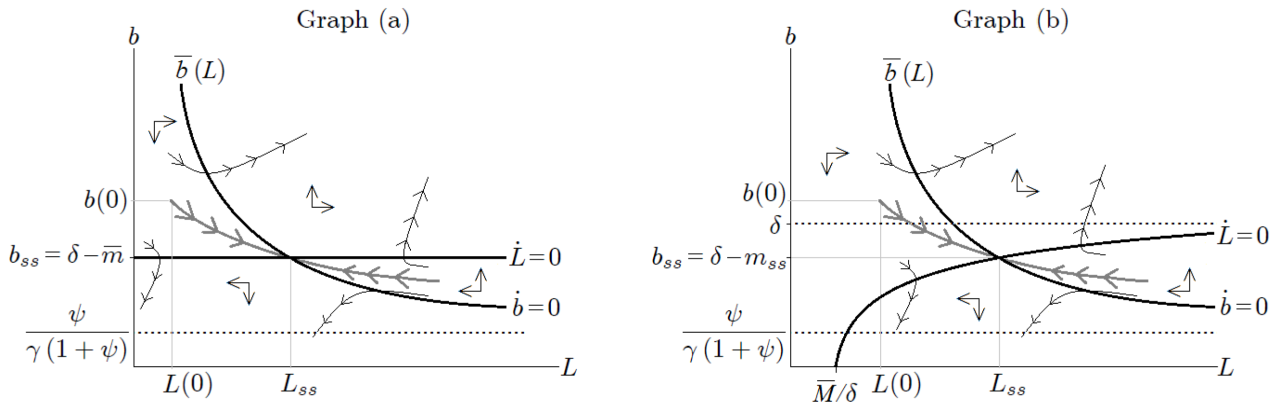

Figure A2.Phase diagrams with migration. Graph (a): constant migration rate $m(t) = \bar{m}$. Graph (b): constant migration level $M(t) = \bar{M}$.

Proof of Proposition 5. Independently of whether the exogenous constant is the migration rate or the migration level, any simultaneous steady state $\dot{L} = \dot{b} = \dot{m} = 0$ characterized by (L_{ss}, b_{ss}, m_{ss}) must obey the general stationarity conditions (D.14)-(D.15) and must fulfill the general existence condition (D.16). The stationarity conditions (D.14)-(D.15) imply the right-hand sides of the steady-state values (35′)-(36′) reported in Proposition 5. The fact that $(L(t), b(t))$ converge asymptotically to such steady-state values (35′)-(36′) follows from Lemma 7 and Lemma 8 above. In particular, from Lemma 7, assuming a constant migration rate implies $m(t) = m_{ss} = \bar{m}$ in each instant and $(L(t), b(t))$ converge to (L_{ss}, b_{ss}) under the stationarity conditions (D.18)-(D.19); from Lemma 8, assuming a constant migration level $M(t) = \bar{M}$ implies the dynamic system (D.20)-(D.21) so that $(L(t), b(t))$ converge to (L_{ss}, b_{ss}) under the stationarity conditions (D.22)-(D.23) and the migration rate converges to $m_{ss} = \bar{M}/L_{ss}$. Note that, combining the steady-state population level (36′) with the equilibrium wealth-wage ratio (28) yields the generalized result

$$\lim_{t\to\infty} \frac{a(t)}{w(t)} = \left(\frac{a}{w}\right)_{ss} \equiv \frac{1}{\eta L_{ss}^{1-\varkappa}} = \frac{2}{\psi} \cdot \frac{\gamma(1+\psi)(\delta - m_{ss}) - \psi}{\rho + \sqrt{\rho^2 + 4\delta(\rho+\delta)\left(1 - \frac{\psi}{\gamma(1+\psi)(\delta - m_{ss})}\right)}}. \quad \text{(D.28)}$$

Derivation of (49) and Proposition 4 with migration. Expressions (D.6), (D.7) and (D.8) are generally valid with or without migration as they hold whenever $(L(t), b(t))$ converge to a simultaneous steady state (L_{ss}, b_{ss}). Substituting $b_{ss} = \delta - m_{ss}$ from (D.14) in the last terms of (D.6), (D.7) and (D.8) we obtain

$$\lim_{t\to\infty}\frac{w(t)\,L(t)\,(1-\gamma b(t))}{A(t)} = \frac{1-\gamma(\delta-m_{ss})}{(a/w)_{ss}}, \tag{D.29}$$

$$\lim_{t\to\infty}\frac{C(t)}{A(t)} = \lim_{t\to\infty}\frac{c(t)}{a(t)} = \left(\frac{c}{a}\right)_{ss} = \frac{\gamma}{\psi}\cdot\frac{(\delta-m_{ss})}{(a/w)_{ss}}, \tag{D.30}$$

$$\lim_{t\to\infty}\frac{w(t)\,L(t)\,(1-\gamma b(t))}{C(t)} = \lim_{t\to\infty}\frac{\frac{w(t)L(t)(1-\gamma b(t))}{A(t)}}{\frac{C(t)}{A(t)}} = \psi\cdot\frac{1-\gamma(\delta-m_{ss})}{\gamma(\delta-m_{ss})}, \tag{D.31}$$

where the last expression (D.31) proves equation (49) in the main text. Note that, in (D.6) and in (D.31), that long-run value $(a/w)_{ss}$ is given by expression (D.28) above.

E Appendix: Quantitative analysis

Calibrating the Model. The target values of endogenous variables and the values of predetermined parameters appearing in Table 1 are calculated as follows.

Population level: the target population $L_{ss} = 36,525,680$, matches the average population of OECD economies in 2015 according to UN (2015) data. The set of countries is: Australia, Austria, Belgium, Canada, Chile, Czech Republic, Denmark, Estonia, Finland, France, Germany, Greece, Hungary, Iceland, Ireland, Israel, Italy, Japan, Korea, Latvia, Luxembourg, Mexico, Netherlands, New Zealand, Norway, Poland, Portugal, Slovak Republic, Slovenia, Spain, Sweden, Switzerland, Turkey, United Kingdom, United States.

Consumption propensity: the target consumption propensity, $c/(a+h) = 0.03$, is the mid-range value of estimated long-run propensities in large OECD countries (Boone and Girouard, 2002) and coincides with traditional benchmark values for the US economy (Poterba, 2000).

Consumption-assets ratio: the target value $(C/A)_{ss} = 0.64$ is calculated from country-level data (OECD, 2017a), where C is identified with Household Final Consumption and A with Financial Net Worth of Households. The longest time series for A cover the 1995-2016 period (with gaps for several countries) and are not available for Mexico and New Zealand. We first calculate the average C/A ratio across OECD countries in each year of the interval 1995-2016, obtaining 22 values ranging from 0.52 to 0.75, and a mean value of 0.64.

Firms-population ratio: The target mass of firms relative to population, $N_{ss}/L_{ss} = 0.0327$, is calculated as follows. First, we calculate the aggregate number of entreprises in 2013 in OECD economies by summing across countries the data reported in OECD (2016a: Ch.2, Table 2.1) obtaining $46,060,568$ firms (all sizes and sectors). Dividing this number by 39 countries, we obtain the OECD average number of firms $N_{ss} = 1,181,040$, which can be divided by the OECD average population $L_{ss} = 36,525,680$ to obtain $N_{ss}/L_{ss} = 0.0327$.

Long-term growth. The target growth rate of final output $(\dot{A}/A)_{ss} = 0.014$ is computed from the long-term GDP forecasts published in OECD (2014a). According to these predictions, the OECD-average real growth rate will converge to 1.4% in year 2060.

R&D propensity. The target value of R&D propensity, 0.022, is the 1995-2015 average of the time series of R&D expenditures-to-GDP in OECD countries reported in OECD (2017b).

Expected adult lifetime. Data from OECD-Stat (2016) for the year 2016 imply an average life expectancy at birth across OECD countries equal to 79.5 years. Since official working-age typically starts between 16 and 18 years, we subtract 17 years to life expectancy at birth, obtaining the average adult lifetime 62.5 that we use in setting the value of $1/\delta$.

Net immigration rate. The reference value $m_{ss} = 0.23\%$ is obtained from OECD (2017c) data reporting 5-year net immigration levels for each country over the 1973-2012 period. We build for each country the estimated yearly net inflows within each 5-year period (as the mean value between consecutive 5-year observations) and then divide the obtained numbers by population in the same year to obtain estimated yearly net immigration rates. We then compute the average net immigration rate for each country over the 1973-2012 period, and then calculate the average across OECD coutries obtaining $m_{ss} = 0.23\%$.

Elasticity of substitution across intermediates. The reference value $\epsilon = 4.3$ is chosen so as to generate in the model an equilibrium mark-up for monopolistic firms $\epsilon/(\epsilon - 1) = 1.3$, which is in the middle of the range 1.2-1.4 suggested by most estimates of economy-wide mark-ups for the US and UK economies – see Britton et al. (2000), Gali et al.

(2007) – and is consistent with OECD estimates by Oliveira Martins et al. (1996).

Gains from variety. As noted in the main text, most of the evidence on the 'gains from variety expansion' comes from empirical studies of international trade. A major reason is that trade data contain detailed records of goods characteristics with highly disaggregated industrial classifications, so that data on internationally traded goods are much more granular than data on goods that are produced and sold domestically. This however implies that the empirical estimates of gains from variety are not fully consistent with the notion of 'gains from variety' typically considered in growth-theoretic models – i.e., the 'gains from differentiation' originally emphasized by Romer (1990). A first problem is that most econometric models of trade identify the gains from variety with a 'love-for-variety' parameter that appears in consumers preferences (Spence, 1976; Dixit and Stiglitz, 1977) whereas our theoretical model follows the standard growth-theoretic specification (Ethier, 1982; Romer, 1990) that identifies such gains from variety with a *technology* parameter. A second problem is that most econometric models of trade estimate *static* gains from variety by treating all traded goods as *final* goods consumed by households: in reality, a large share of traded goods consists of intermediate inputs/capital goods and this implies that the actual gains from variety should include non-negligible dynamics gains that are however neglected in empirical studies. A third problem is that studies of trade typically concentrate on the welfare gains that an economy enjoys as a result of increases in the mass of *imported* varieties: if there exists a non-negligible difference between the number of varieties that a country imports from abroad and the number of varieties that the country produces and sells domestically, the overall 'gains from variety expansion' that an economy enjoys may substantially differ from the gains from imported varieties. The three problems mentioned above suggest a flexible parametrization of $\chi - 1$, the elasticity of productivity to the mass of intermediate goods. The closest empirical counterpart to our $\chi - 1$ is the 'elasticity of productivity to variety' calculated by Broda et al. (2006), which ranges from 0.05 to 0.20. We adopt the baseline value $\chi - 1 = 0.05$ and subsequently check the sensitivity of our results to imposing the alternative values 0.10-0.15.

Appendix: References

Barro, R., Sala-i-Martin, X. (2004). *Economic Growth.* MIT Press.

Blanchard, O. (1985). Debt, Deficits, and Finite Horizons. *Journal of Political Economy* 93: 223-247.

Boone, L., Girouard, N. (2002). The stock market, the housing market and consumer behaviour. *OECD Economic Studies* 35, 2002/2.

Britton, E., Larsen, J., Small, I. (2000). Imperfect competition and the dynamics of markups. *Bank of England working papers* 110, Bank of England.

Broda, C., Greenfield, J., Weinstein, D. (2006). From Groundnuts to Globalization: A Structural Estimate of Trade and Growth. *NBER Working Papers* 12512, National Bureau of Economic Research.

Dixit, A., Stiglitz, J. (1977). Monopolistic Competition and Optimum Product Diversity. *American Economic Review* 67 (3): 297-308.

Ethier, W. (1982). National and International Returns to Scale in the Modern Theory of International Trade. *American Economic Review* 72 (3): 389-405.

Gali, J., Gertler, M., López-Salido, J.D. (2007). Markups, Gaps, and the Welfare Costs of Business Fluctuations. *Review of Economics and Statistics* 89 (1): 44-59.

OECD (2014a). *Economic Outlook No 95 - May 2014*, www.stats.oecd.org (dataset accessed July 2017). Organisation for Economic Co-operation and Development.

OECD (2016a). *Entrepreneurship at a Glance 2016.* Organisation for Economic Co-operation and Development.

OECD (2017a). *National Accounts at a Glance 2017*, www.stats.oecd.org (dataset accessed July 2017). Organisation for Economic Co-operation and Development.

OECD (2017b). *Research and Development Statistics 2017*, www.stats.oecd.org (dataset accessed July 2017). Organisation for Economic Co-operation and Development.

OECD (2017c). *World Development Indicators*, www.stats.oecd.org (dataset accessed July 2017). Organisation for Economic Co-operation and Development.

Oliveira Martins, J., Scarpetta, S., Pilat, D. (1996). Mark-up pricing, market structure and the business cycle. *OECD Economic Studies* 27, 1996/II.

Poterba, J.M. (2000). Stock Market Wealth and Consumption. *Journal of Economic Perspectives* 14 (2): 99–118.

Romer, P. (1986). Increasing Returns and Long-run Growth. *Journal of Political Economy* 94 (5): 1002-1037.

Romer, P. (1990). Endogenous Technological Change. *Journal of Political Economy* 98 (5): 71-102.

Spence, M. (1976). Product Differentiation and Welfare. *American Economic Review* 66 (2): 407-414.

UN (2015). *World Population Prospects: The 2015 Revision, DVD Edition.* United Nations, Department of Economic and Social Affairs, Population Division.

World Bank (2017). *Databank – crude brith rates*, downloaded from https://data.worldbank.org (dataset accessed July 2017).