The Attack-and-Defense Group Contests*

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Abstract

This study analyzes a group contest in which one group (defenders) follows a weakest-link whereas the other group (attackers) follows a best-shot impact function. We fully characterize Nash and coalition-proof equilibria and show that with symmetric valuation the coalition-proof equilibrium is unique up to the permutation of the identity of the active player in the attacker group. With asymmetric valuation it is always an equilibrium for one of the highest valuation players to be active; it may also be the case that the highest valuation players in group 1 free-ride completely on a player with a lower valuation. However, in any equilibrium, only one player in the attacker group is active, whereas all the players in the defender group are active and exert the same effort. We also characterize Nash and coalition-proof equilibria for the case in which one group follows either a best-shot or a weakest-link but the other group follows an additive impact function.

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1. Introduction

Consider a situation in which a group of firms are engaged in illegal price fixing. Their businesses are spread over several countries and each of them exerts irreversible resources on hiding their activities and on legal experts to avoid possible prosecution (Malik, 1990). Anti-trust authorities in those countries (the Office of Fair Trading in the UK, and the Anti-trust division of the Department of Justice in the USA, for example) also exert costly resources on investigation to detect possible cartels. Thus, we can depict the anti-trust authorities as a group of 'attackers' and the colluding firms as a group of 'defenders'. For the anti-trust authorities, the efforts have the nature of a 'best-shot', i.e., if any of the authorities can detect the cartel, then it will solve the problem for all others. Hence, essentially the best effort exerted among the authorities represents the strength of the investigation. However, for the colluding firms, the resources exerted have the nature of a weakest-link, i.e., if one of them gets detected, then the whole cartel will be detected. Therefore, the lowest avoidance effort determines the strength of hiding the collusion.[1]

The above mentioned situation can be structured as a group contest, in which members of groups exert irreversible efforts that translate into 'group effort'. Then a group contest success function (a function that maps the group efforts into the probabilities of winning) determines which group is going to win. A function that translates the individual group member efforts into a group effort is called an impact function (Wärneryd, 1998). In the case above, the colluding firms follow a weakest-link impact function, and the anti-trust authorities follow a best-shot impact function. We term this type of game as the 'Attack-and-Defense Group Contest'.

This type of structure is quite common in the field. In addition to the case of avoidance efforts in collusion discussed above, there are very many situations in which groups are engaged in Attack and Defense Contests. One prominent example is from defense economics (Conybeare et al., 1994; Arce et al., 2012). The siege game between different intelligence agencies and terrorist organizations follow this structure. Intelligence agencies such as the CIA or the FBI trying to stop terrorists follow a best-shot impact function, since if any of them can capture or uncover a terrorist ploy, it will solve the issue. However, terrorist organizations such as the Al-Quida or the Lashkar-e-Taiba follow a weakest-link impact function, since if any of their ploys

[1] In some particular circumstances, the strength of either the attackers or the defenders can arguably be viewed also as perfectly substitutable or additive in nature. We discuss them in detail in Section 3.

get detected, then the terror links will be exposed.[2] A further example comes from the system reliability (Varian, 2004; Hausken, 2008) literature. A system of operations follows a weakest-link structure for if one of them is captured by viruses, then the whole system will get infected. However, for virus codes, it is a best-shot situation, since if any of the viruses can get in through the system, then it can infect and capture the whole system. In corrupt societies the members of a political party exert efforts to conceal information regarding corruption in the party, whereas members of the civil society exert effort to uncover it (OECD, 2003). Understandably, the strength of the party will be as strong as the weakest member, but for the civil society any successful member will bring in the required result for all.[3]

Individual attack and defense game is explored extensively in the literature. Clark and Konrad (2007) analyze a game in which multiple battlefields are linked and two players, with limited resources, allocate the resources into the battlefields. The resources allocated by both players in a particular battlefield determine the probability of a player to win the battlefield. One player, the attacker, will have to win at least one battlefield to win the game; but the other player, the defender, will have to win every battlefield. This type of structure was later employed in other studies (Hausken, 2008; Arce et al., 2012; Kovenock and Roberson, 2012) and generalized in a model by Kovenock and Roberson (2010). Each study, however, analyzes individual conflicts and does not consider any group dynamics in this context.

The literature on group contests starts with the work by Katz et al. (1990) who considered symmetric valuation and a lottery (Tullock, 1980) contest success function with a perfectly substitute (additive) impact function. Baik (2008) extended this to asymmetric valuation and showed that only the highest valuation player in a group exerts effort in equilibrium whereas other players free-ride. Lee (2012) employs the weakest-link impact function for all groups and characterizes possible equilibria. He shows that in any equilibrium all group members of a group exert the same effort. Multiple equilibria exist, but the Pareto efficient equilibrium is unique. Kolmar and Rommeswinkel (2013) use a CES impact function (ranging from perfect substitute

[2] On the other hand, when the roles reverse, i.e., terrorist groups attack a country, then any successful attack serves their purposes and hence they follow a best-shot technology. However, the intelligence agencies now are in the defense positions and they lose if any successful attack occurs. Hence they follow a weakest-link technology.

[3] Of course, this structure can also capture situations beyond the nature of attack and defense. For example, a firm in a patent race may run parallel R&D teams, but another firm may run a big R&D team that works sequentially by specialized team members (Nelson, 1961; Abernathy and Rosenbloom, 1968). Hence, the resultant R&D of the first firm in the patent race will have the nature of a best-shot since the best product will represent the firm. However, the resultant R&D of the second firm will depend on the strength of the weakest member.

to weakest-link) and characterize the set of equilibria. Chowdhury et al. (2013a) employ the best-shot impact function and show that only one player in each active group exerts positive effort in equilibria. However, the active player need not be a highest valuation group member. All these studies employ a stochastic (lottery, *a la* Tullock, 1980) contest success function. Another stream of research, instead, uses a deterministic (all-pay auction, *a la* Baye et al., 1996) contest success function. Among them, Baik et al. (2001) and Topolyan (2013) use an additive impact function; whereas Chowdhury et al. (2013b) and Barbieri et al. (2013) use weakest-link and best-shot technologies, respectively.

These studies make several important contributions to the literature. However, none of them explore a situation in which different groups can follow different impact functions. Table 1 summarizes the fit of the current study in this area of literature.

Table 1. Fit of the current study.

<table>
<tr><th>Impact Func
CSF</th><th>Best Shot</th><th>Perfect Substitute</th><th>Weakest Link</th></tr>
<tr><td rowspan="3">Stochastic (Tullock)</td><td rowspan="2">Chowdhury et al. (2013a)</td><td>Katz et al. (1990), Baik (2008)</td><td>Lee (2012)</td></tr>
<tr><td colspan="2">Kolmar and Rommeswinkel (2013): CES from perfect substitute to weakest link</td></tr>
<tr><td colspan="3">Current study: Allows the possibility of different impact functions for different groups</td></tr>
<tr><td>Deterministic (All-pay auction)</td><td>Barbieri et al. (2013)</td><td>Baik et al. (2001), Topolyan (2013)</td><td>Chowdhury et al. (2013b)</td></tr>
</table>

To summarize, in this study we make a three-fold contribution. First, we provide a better understanding of situations in which groups are engaged in attack and defense conflicts. For the first time in the literature, we introduce a theoretical underpinning of attack and defense contests in which groups, rather than individuals, are involved. Second, we fill in a gap in the group contest literature by providing with contests in which different groups follow different impact functions. Third, we consider the prize to have the nature of group-specific public good (e.g., if the colluding firms are not detected, then every firm is benefitted; and if they are detected then every anti-trust authority is). Hence, we also contribute to the literature of collective action

(Olson, 1965) and public good game (Bliss and Nalebuff, 1984; Bergstrom et al., 1986; Barbieri and Malueg, 2008) with weakest-link or best-shot network externalities (Hirshleifer, 1983, 1985; Cornes, 1993).

2. The Model

2.1. Model Set-up

We structure the model along similar lines to Chowdhury et al. (2013a). Consider a contest in which 2 groups compete to win a group-specific public-good prize. Group g consists of $m_g \geq 2$ risk-neutral players who exert costly efforts to win the prize. The individual group members' valuation for the prize may differ across groups; however it is the same within a group. Let $v_g > 0$ represent the valuation for the prize of any player in group g. Let $x_{gi} \geq 0$, measured in the same unit as the prize values, represent the effort level exerted by player i in group g.

Next we specify the *group impact function* as $f_g\colon \mathbb{R}_+^{m_g} \to \mathbb{R}_+$, such that the group effort of group g is given by $X_g = f_g(x_{g1}, x_{g2}, \dots, x_{gm_g})$. The following assumptions define the best-shot technology for group 1 and the weakest-link technology for group 2.

Assumption 1. The group effort of group 1 is represented by the maximum effort level exerted by the players in group 1, i.e., $X_1 = max\{x_{11},\ x_{12}, \dots, x_{1m_1}\}$.

Assumption 2. The group effort of group 2 is represented by the minimum effort level exerted by the players in group 2, i.e., $X_2 = min\{x_{21},\ x_{22}, \dots, x_{2m_2}\}$.

To specify the winning probability of group g, denote $p_g(X_1, X_2)\colon \mathbb{R}_+^2 \to [0,1]$ as a *contest success function* (CSF). We assume a logit form group CSF (Münster, 2009).

Assumption 3. The probability of winning the prize for group g is

$$p_g(X_1, X_2) = \begin{cases} X_g/(X_1 + X_2) & \text{if } X_1 + X_2 > 0 \\ 1/2 & \text{if } X_1 + X_2 = 0 \end{cases}$$

We assume all players forgo their efforts and they have a common cost function with unit marginal cost as described by *Assumption 4*.

Assumption 4. The common cost function is $c(x_{gi}) = x_{gi}$.

Only the members of the winning group receive the prize. Let π_{gi} represent the payoff for player i in group g. Under the above assumptions, the payoff for player i in group g is:

$$\pi_{gi} = v_g \frac{X_g}{X_1+X_2} - x_{gi}. \quad (1)$$

Equation (1) along with the four assumptions represents the *attack-and-defense group contest*. To close the structure we assume that all players in the contest choose their effort levels independently and simultaneously, and that all of the above (including the valuations, group compositions, impact functions, and the contest success function) is common knowledge. We employ Nash equilibrium as our solution concept.

We use the following definitions throughout the paper.

Definition 1. If player i in group g exerts strictly positive effort, i.e., $x_{gi} > 0$, then the player is called *active*. Otherwise (when $x_{gi} = 0$) the player is called *inactive*.

Definition 2. If the group effort of group g is strictly positive, i.e., $X_g > 0$, then group g is called *active*. Otherwise (when $X_g = 0$) the group is called *inactive*.

Definition 3. Let I_g denote the index set of all players in group g, and $\mathcal{P}_g$ denote the set of all nonempty subsets of I_g. We say that a coalition of players $C \in \mathcal{P}_g$ of group g *blocks* a strategy profile $x = (x_{11}, \ldots, x_{1m_1}, x_{21}, \ldots, x_{2m_2})$ if there exists a strategy profile y such that $u_{g,i}(y) \geq u_{g,i}(x)$ for all $i \in C$, $u_{g,j}(y) > u_{g,j}(x)$ for some $j \in C$, $x_{-g,i} = y_{-g,i}$ for all $i \in I_{-g}$, and $x_{g,k} = y_{g,k}$ for all $k \notin C$.

In other words, a coalition of players blocks a strategy profile if the members of the coalition have an incentive to deviate *altogether*.

Definition 4. A strategy profile $\boldsymbol{x}$ is called a *coalition-proof equilibrium* if no coalition $C \in \mathcal{P}_g$, $g = 1, 2$, blocks $\boldsymbol{x}$.

It follows from *Definition 4* that any coalition-proof equilibrium is a Nash equilibrium.

2.2. Solution with Symmetric Valuations

We begin by stating *Lemma 1*. This lemma points out (from Assumption 3) that both groups actively participate in the contest.

Lemma 1. *In any equilibrium both groups are active.*

Assumption *1* gives rise to *Lemma 2*. This result is analogous to *Lemma 2* of Chowdhury et. al. (2013a), and the proof follows similar lines.

Lemma 2. *In any equilibrium only one player in group 1 is active.*

The following result in *Lemma 3* is analogous to *Lemma 1* of Lee (2012) and holds due to the weakest-link technology in group 2.

Lemma 3. *In any equilibrium all players of group 2 are active and choose the same effort level.*

Lemmata 1, 2, and *3* simplify the group contest into a seemingly individual contest in which each group behaves like an individual, but the valuation of the individual contestant may change depending on which group member within group 1 is active. Consider a situation where only player i of group 1 is active and puts effort x_1 ("attacks") and all players of group 2 "defend", exerting the same effort level x_2.. Note that because of the best-shot effort technology in group 1 and the weakest-link technology in group 2, we have $X_1 = x_1$ and $X_2 = x_2$. Equation (1) yields the following first-order conditions for an interior Nash equilibrium.

$$v_1 x_2 / (x_1 + x_2)^2 = 1 \tag{2}$$

$$v_2 x_1 / (x_1 + x_2)^2 \geq 1 \tag{3}$$

Equation (2) results from the fact that the active player of group 1 is competing individually against group 2. Inequality (3) ensures that no player in group 2 wants to decrease her effort level (due to the weakest-link technology no player in group 2 wants to deviate to a higher effort level). Equation (2) implies $x_1 = \sqrt{v_1 x_2} - x_2$; plug this to (3) to get

$$x_2 \leq (v_1 v_2^2)/(v_1 + v_2)^2 \equiv \bar{x}_2$$

Therefore, there exists a continuum of equilibria such that only one player in group 1 exerts effort $x_1 = \sqrt{v_1 x_2} - x_2$ and every player in group 2 exerts effort $x_2 \in (0, \bar{x}_2]$.

Since the weakest effort determines the survival of the defenders, all members put forth the same effort in equilibrium. However, since the highest effort determines the success of the attacking group, its members have an incentive to free ride – and in equilibrium only one of the group members exerts effort and all others free ride on him. Note that though there exists a continuum of Nash equilibria, there is a unique coalition-proof equilibrium where all players in group 2 choose the highest effort level $\bar{x}_2$. Theorem 1 summarizes this result.

Theorem 1. *The attack and defense contest with symmetric valuation has a unique coalition-proof equilibrium (up to the permutation of the identity of the active player in group 1), where one player in group 1 exerts effort* $x_1 = (v_1^2 v_2)/(v_1 + v_2)^2$*, every other player in group 1 puts zero effort, and every player in group 2 exerts effort* $x_2 = (v_1 v_2^2)/(v_1 + v_2)^2$*. There is a continuum of Nash equilibria such that every player in group 2 exerts effort* $0 < x_2 \leq (v_1 v_2^2)/(v_1 + v_2)^2$*, one player in group 1 exerts effort* $x_1 = \sqrt{v_1 x_2} - x_2$*, and all other players in group 1 exert zero effort.*

2.3. Extension to Asymmetric Valuations

Assume now that the individual group members' valuation for the prize may differ within and across groups. This asymmetry in values can reflect player asymmetry, or an exogenous sharing rule of the group-specific prize, in which the prize-shares among the members of a group are different. Also, note that if we relax Assumption 4 and consider player asymmetry in marginal costs then, due to risk neutrality, the model can again be transformed into an asymmetric valuation model after appropriate rescaling. Let $v_{gi} > 0$ represent the valuation for the prize of player i in group g. Without loss of generality, assume $v_{g(t-1)} \geq v_{gt}$ for $m_g \geq t > 1$.

Here we cannot apply the method we used in the previous section to find Nash equilibria because, given the heterogeneity of valuations in the defender team, it is not clear which valuation to use to derive the first-order conditions. We can, however, simplify the game as follows. Assume all players in group 2 choose the same effort level (either in or off equilibrium). Clearly, no player wants to deviate to a higher effort due to the weakest-link technology. It may be the case, however, that some player wants to exert a lower effort. Thus, a strategy $(x_{21},$

$\cdots,\ x_{2m_2})$ is a part of equilibrium if and only if $x_{21} = x_{22} = \cdots = x_{2m_2}$ and no player in group 2 wants to deviate to some lower effort.

Clearly, only one player in group 1 is active due to the best-shot impact function. Suppose one of the highest valuation players (say, player 1) is active and exerts effort x_{11}. Due to the weakest-link technology all players in group 2 exert the same effort level x_2. As in the previous section, player 1 in group 1 is competing individually against group 2, which yields the following first-order condition.

$$v_{11}x_2/(x_{11} + x_2)^2 = 1 \tag{4}$$

To ensure that no player in group 2 wants to deviate, the following system of inequalities must be satisfied.

$$\begin{cases} v_{21}x_{11}/(x_{11} + x_2)^2 \geq 1 \\ \vdots \\ v_{2m_2}x_{11}/(x_{11} + x_2)^2 \geq 1 \end{cases}$$

Since the valuations are descending within the group, this system is equivalent to

$$v_{2m_2}x_{11}/(x_{11} + x_2)^2 \geq 1 \tag{5}$$

Denote by x_{1k}^b the best response of player *k* of group 1 under the condition that player *k* puts the highest effort in her group (not necessarily in equilibrium), then clearly $v_{1k}x_2/\left(x_{1k}^b + x_2\right)^2 \leq 1$ for all $k > 1$ given that $v_{1k}x_2/\left(x_{1k}^b + x_2\right)^2$=1. Therefore $x_{11}^b \geq x_{1k}^b$, which shows that when *only* player 1 in group 1 is active, no other (inactive) player in group 1 wants to deviate.

Hence Equations (4) and (5) characterize all Nash equilibria in which one of the highest valuation players is active. This system of equations has a continuum of solutions which are of the following form: $x_{11} = \sqrt{v_{11}x_2} - x_2$ and $x_2 \in (0, \bar{x}_2]$, where

$$\bar{x}_2 = (v_{11}v_{2m_2}^2)/(v_{11} + v_{2m_2})^2 \tag{6}$$

Let us now investigate the possibility that the highest valuation players free-ride on a player with a lower valuation. Suppose only player *k* (such that $v_{1k} < v_{11}$) in group 1 is active, then he chooses effort level $x_{1k} = \sqrt{v_{1k}x_2} - x_2$. Therefore an inactive player *j* earns payoff

$$\pi_{1j}^{\text{free-ride}} = \frac{\sqrt{v_{1k}} - \sqrt{x_2}}{\sqrt{v_{1k}}} v_{1j}$$

Let us investigate whether player *j* has a profitable deviation. Since player *j* is trying to maximize her payoff, she would deviate, if at all, to the effort level x_{1j} such that $x_{1j} = \sqrt{v_{1j}x_2} - x_2$, which is implied by the corresponding first-order condition. Since $v_{1(t-1)} \geq v_{1t}$ for $m_g \geq t > 1$, we have $x_{1(t-1)} \geq x_{1t}$. Consequently, no player *t* in group 1 such that $t > k$ has an incentive to deviate, for player *t* will not be the best-shot in her group if she exerts effort level x_{1t}. Fix player $j < k$ who exerts effort $x_{1j} = \sqrt{v_{1j}x_2} - x_2$, then player *j*'s payoff is

$$\pi_{1j}^{\text{active}} = \frac{\sqrt{v_{1j}} - \sqrt{x_2}}{\sqrt{v_{1j}}} v_{1j} - \sqrt{v_{1j}x_2} + x_2$$

Hence player *j* has no incentive to become active if and only if $\pi_{1j}^{\text{free-ride}} \geq \pi_{1j}^{\text{active}}$, i.e.,

$$\frac{\sqrt{v_{1k}} - \sqrt{x_2}}{\sqrt{v_{1k}}} v_{1j} \geq \frac{\sqrt{v_{1j}} - \sqrt{x_2}}{\sqrt{v_{1j}}} v_{1j} - \sqrt{v_{1j}x_2} + x_2,$$

which is equivalent to

$$2\sqrt{v_{1j}v_{1k}} - v_{1j} \geq \sqrt{x_2 v_{1k}} \tag{7}$$

Note that the for any fixed v_{1k}, the right-hand side of (7) approaches zero as $x_2 \to 0$. Thus, condition (7) is satisfied for some $x_2 > 0$ if and only if $2\sqrt{v_{1j}v_{1k}} - v_{1j} > 0$, which is equivalent to

$$4v_{1k} \geq v_{1j} \tag{8}$$

Note also that if condition (7) holds for some $x_2 = r$, then it holds for all $0 < x_2 \leq r$. Fix player k in group 1 such that $v_{1k} < v_{11}$, and for each player $j < k$ define

$$r_{jk} = sup\{0 \leq y \leq \bar{x}_2 : 2\sqrt{v_{1j}v_{1k}} - v_{1j} \geq \sqrt{y v_{1k}}\},$$

where $\bar{x}_2 = (v_{1k}v_{2m_2}^2)/(v_{1k} + v_{2m_2})^2$.

By convention we let $sup\{\emptyset\} = -\infty$. Next, define

$$r_k = \min_{1\leq j<k} r_{jk} \tag{9}$$

By construction, $r_k > 0$ if and only if $4v_{1k} \geq v_{1j}$ for all $j < k$. We are now ready to characterize all Nash equilibria for the case of asymmetric valuation.

Theorem 2. The Nash equilibria of *the attack-and-defense contest with asymmetric valuation are as follows.*

1. *If for every player k such that $v_{1k} < v_{11}$ there exists player $j < k$* such that $4v_{1k} < v_{1j}$, *then there exists a continuum of Nash equilibria such that all players in group 2 are active and exert effort $x_2 \in (0, \bar{x}_2]$ while only one of the highest valuation players in group 1 is active and exerts effort $x_1 = \sqrt{v_{11}x_2} - x_2$,* where $\bar{x}_2 = (v_{11}v_{2m_2}^2)/(v_{11} + v_{2m_2})^2$.
2. *If $4v_{1k} \geq v_{1j}$ for some k such that $v_{1k} < v_{11}$ and all $j < k$, then in addition there exists a continuum of equilibria such that that all players in group 2 are active and exert effort $x_2 \in (0, r_k]$, where r_k is defined by Equation (9), player k in group 1 is active and exerts effort $x_{1k} = \sqrt{v_{1k}x_2} - x_2$,* and all other players in group 1 put no effort.

Finally, let us investigate the existence of coalition-proof equilibria in which the highest-valuation players may be inactive. Suppose only player *k* in group 1 is active (where $v_{1k} < v_{11}$), then there is a unique candidate for the coalition-proof equilibrium (note that $x_{1j} = 0$ for all $j \neq k$).

$$x_{1k} = (v_{1k}^2 v_{2m_2})/(v_{1k} + v_{2m_2})^2 \tag{10}$$

$$x_2 = (v_{1k}v_{2m_2}^2)/(v_{1k} + v_{2m_2})^2 \tag{11}$$

Consequently player j in group 1, who puts no effort, earns payoff

$$\pi_{1j}^{\text{free-ride}} = \frac{v_{1k}^2 v_{2m_2}}{[v_{1k}^2 v_{2m_2} + v_{1k}v_{2m_2}^2]} v_{1j} = \frac{v_{1k}}{[v_{1k} + v_{2m_2}]} v_{1j}$$

Since the valuations are descending within the group, we again conclude that no player $j > k$ wants to become active. Fix player $j < k$; her best option for a deviation is

$$x_{1j} = (v_{1j}^2 v_{2m_2})/(v_{1j} + v_{2m_2})^2,$$

in which case her payoff is

$$\pi_{1j}^{\text{active}} = \frac{v_{1j}^2 v_{2m_2}}{\left[v_{1j}^2 v_{2m_2} + v_{1j} v_{2m_2}^2\right]} v_{1j} - v_{1j}^2 v_{2m_2}/\left(v_{1j} + v_{2m_2}\right)^2$$

Hence, player j will free-ride on player k if and only if $\pi_{1j}^{\text{free-ride}} \geq \pi_{1j}^{\text{active}}$, which is equivalent to

$$v_{1k} \geq v_{1j}^2/(v_{2m_2} + v_{1j}) \tag{12}$$

Rewrite condition (12) as $v_{1j}\left(v_{1k} - v_{1j}\right) + v_2 v_{1k} \geq 0$, from which it is evident (since $v_{1j} \geq v_{1k}$) that if condition (12) is satisfied for player 1 in group 1, then it is satisfied for every $j < k$. Thus all players in group 1 except for player k are better-off exerting no effort if and only if

$$v_{1k} \geq v_{11}^2/(v_{2m_2} + v_{11}) \tag{13}$$

Therefore the coalition-proof equilibrium where only player k in group 1 is active exists if and only if condition (13) is satisfied; such equilibrium is described by equations (10) and (11). Note that there always exists a coalition-proof equilibrium in which a highest valuation player is active. If at least two players in group 1 tie for the highest valuation, then the coalition-proof equilibrium effort levels where a highest valuation player in group 1 is active are unique while any one of the highest valuation players is active.

Theorem 3. The coalition-proof equilibria of *the attack-and-defense contest with asymmetric valuation are as follows.*

1. *If* $v_{1k} < \frac{v_{11}^2}{v_{2m_2}+v_{11}}$ *for all* k *such that* $v_{1k} < v_{11}$ *, then there is a unique coalition-proof equilibrium (up to the permutation of the identity of a highest valuation player) in which one of the highest valuation players in group 1 is active and exerts effort* $x_1 = (v_{11}^2 v_{2m_2})/(v_{11} + v_{2m_2})^2$*, while every player in group 2 exerts effort* $x_2 = (v_{11} v_{2m_2}^2)/(v_{11} + v_{2m_2})^2$.

2. *If* $v_{1k} \geq v_{11}^2/(v_{2m_2} + 2v_{11})$ *for some k>1, then there exists a coalition-proof equilibrium where only player k in group 1 is active and exerts effort* $x_{1k} = (v_{1k}^2 v_{2m_2})/(v_{1k} + v_{2m_2})^2$*, while every player in group 2 exerts effort* $x_2 = (v_{1k} v_{2m_2}^2)/(v_{1k} + v_{2m_2})^2$.

3. Cases with an Additive Impact Function

It can be argued that the attack effort does not necessarily need to follow a best-shot function. To be precise, it may be possible that the effort of one member of the attacker group can add up to (or substitute to) another member's effort. Examples of this may be situations in which the attackers such as the CIA or the FBI make mutually exclusive geographical or jurisdictional restrictions. In such a case the attacker group follows a perfectly substitute impact function. To incorporate this structure, we keep all the other assumptions in the model the same but replace Assumption 1 with

Assumption 1'. The group effort of group 1 is represented by the sum of effort levels exerted by the players in group 1, i.e., $X_1 = \sum_{i=1}^{m_1} x_{1i}$.

Similar to the analyses above, all the players in group 2 exert the same effort. Following Baik (2008), the equilibrium effort of group 1 is unique. Only the highest value player(s) in group 1 exert positive effort whereas all other group members exert zero effort. If there is more than one player with the highest valuation, then the total equilibrium effort of group 1 can be determined, but not the individual efforts. This is summarized in Theorem 4.

Theorem 4. T*he attack and defense contest under Assumptions 1', 2, 3, and 4 has a continuum of equilibria. There may be multiple equilibria corresponding to the same effort level of group 2, depending on which players in group 1 are active. In equilibrium every player in group 2 exerts effort* $0 < x_2 \leq (v_{11} v_{2m_2}^2)/(v_{11} + v_{2m_2})^2$*, the highest valuation players in group 1 exert collective effort of* $x_1 = (v_{11}^2 v_{2m_2})/(v_{11} + v_{2m_2})^2$*, while all other group members in group 1 exert zero effort.*

There may exist multiple coalition-proof equilibria. All of them, however, induce the same group efforts. In any coalition-proof equilibrium the highest valuation players in group 1 exert *collective effort of* $x_1 = (v_{11}^2 v_{2m_2})/(v_{11} + v_{2m_2})^2$*, each player in group 2 exerts effort* $x_2 = (v_{11} v_{2m_2}^2)/(v_{11} + v_{2m_2})^2$*, and all other players in group 1 put no effort.*

Proof. Clearly, in any equilibrium, all players of group 2 exert the same effort level x_2 due to the weakest-link technology. Let I_1 and I_{1H} denote the index sets of all players in group 1 and the highest valuation players in group 1, respectively. Fix player $k \in I_{1H}$, and suppose by contradiction there exists a player $j \in I_1 \backslash I_{1H}$ whose equilibrium effort level is $x_{1j} > 0$. As before, denote by X_1 equilibrium group of group 1. Then the first-order conditions imply

$$v_{1k} x_2/(X_1 + x_2)^2 = 1$$

$$v_{1j} x_2/(X_1 + x_2)^2 = 1$$

This leads to a contradiction since $v_{1k} > v_{1j}$. Therefore, only the highest valuation players in group 1 exert a positive effort and $\frac{v_{1j} x_2}{(X_1 + x_2)^2} < 1$ for all $j \in I_1 \backslash I_{1H}$. The following system characterizes all equilibria (along with $x_{1j} = 0$ for all $j \in I_1 \backslash I_{1H}$).

$$v_{11} x_2/(X_1 + x_2)^2 = 1$$

$$v_{2m_2} X_1/(X_1 + x_2)^2 \geq 1$$

Therefore any strategy profile, such that every player in group 2 exerts effort $0 < x_2 \leq (v_{11} v_{2m_2}^2)/(v_{11} + v_{2m_2})^2$, $\sum_{i \in I_{1H}} x_{1i} = \sqrt{v_{11} x_2} - x_2$ and $x_{1j} = 0$ for all $j \in I_1 \backslash I_{1H}$ in group 1, is an equilibrium. To derive coalition-proof equilibria, one needs to solve the system

$$v_{11} x_2/(X_1 + x_2)^2 = 1$$

$$v_{2m_2} X_1/(X_1 + x_2)^2 = 1$$

Note that all the coalition-proof equilibria result in the same outcome with respect to the group efforts: $(v_{11}^2 v_{2m_2})/(v_{11} + v_{2m_2})^2$ and $(v_{11} v_{2m_2}^2)/(v_{11} + v_{2m_2})^2$ in groups 1 and 2, respectively. This completes the proof. ■

Next, it may also be argued that instead of following a weakest-link impact function, the defenders follow an additive impact function. To incorporate this, we keep all the other assumptions unaltered but assumption 2 is replaced with:

Assumption 2'. The group effort of group 2 is represented by the sum of effort levels exerted by the players in group 2, i.e., $X_2 = \sum_{i=1}^{m_2} x_{2i}$.

Again, following the analysis in Theorem 2, it is easy to show that only one player in group 1 is active in equilibrium. Multiple equilibria exist, and the efforts depend on the identity of the active player in group 1. Once again, similar to Baik (2008), the equilibrium effort of group 2 is unique. Only the highest value player(s) in group 2 exert positive effort and all other group members exert zero effort. If there is more than one player with the highest valuation, then the total equilibrium effort of group 2 can be determined, but not the individual efforts. This result is summarized in Theorem 5; the proof is similar to the ones of Theorems 3 and 4.

Theorem 5. *The attack and defense contest under Assumptions 1, 2', 3, and 4 has up to m_1 equilibria. In any equilibrium there is only one active player from group 1. The condition for only player k in group 1 to be active in equilibrium is $v_{1k} \geq v_{11}^2/(v_{21} + v_{11})$. If player k in group 1 is active, then the highest valuation players in group 2 exert collective effort of $x_2 = (v_{1k}v_{21}^2)/(v_{1k} + v_{21})^2$, and player k from group 1 exerts $x_{1k} = (v_{1k}^2 v_{21})/(v_{1k} + v_{21})^2$ while all other players from either group exert zero effort. Coalition-proof equilibria are the same as the Nash equilibria.*

4. Discussion

We analyze a group contest in which one group follows a best-shot and the other group follows a weakest-link impact function. This setting may be viewed as a stylized representation of situations in which one group attacks and the best effort out of the group members determines the strength of the attack, whereas the other group defends and the weakest effort among the group members represents the strength of the defense. This study adds to the attack and defense literature since it introduces a group setting in this area of literature for the first time. It also introduces hybrid impact functions (different groups with different impact functions) in the group contest literature for the first time.

We fully characterize Nash and coalition-proof equilibria and show that under symmetric valuation the game has a unique coalition-proof equilibrium up to the permutation of the identity of the active player in the attacker group. When the valuations are asymmetric, a wider variety of equilibria is possible. It is always an equilibrium for one of the highest valuation players to be active, but it may also be possible that the active player does not have the highest valuation. In any equilibrium, only one player in the attacker group is active, whereas all the players in the defender group are active and exert the same effort. We also characterize Nash and coalition-proof equilibria for the case in which one group follows a perfectly substitute impact function whereas the other group follows either a best-shot or a weakest-link impact function.

The results imply that all the defenders (the colluding firms or the terrorist group members), due to the perfectly complementary nature of their collective action; participate in defense activity without free riding. However, they exert resources only according to the strength of the weakest member of the group. If the attackers (the anti-trust or security authorities) follow a perfectly substitute collective action then all the weaker or less efficient group members free ride on the strongest one. However, if the collective action has the nature of a best-shot, and the valuation or efficiency of the strongest group member is not relatively too high, then other group members may also exert resources.

There are several possible ways – both in theory and in application – to extend the current analysis. First, we have employed a stochastic contest success function (Tullock, 1980), but in some cases the contests are deterministic and it is more appropriate to apply an all-pay auction contest success function to solve the problem. It may also be possible to employ a generic CES impact function and vary the elasticity of substitution across groups to achieve a very general solution in this area of investigation. The analyses can be extended to more than two groups and with the employment of more than two impact functions. Finally, it is also possible to test the predictions of the theory and investigate whether any particular equilibrium is focal in the field by implementing them in the laboratory (Sheremeta, 2011), or whether within and across groups design tools such as punishment (Abbink et al., 2010) or communication (Cason et al., 2012) affect subject behavior. However, each of these issues is beyond the scope of the current study and we leave them for future research.

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