Dynamic Multilateral Markets☆

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Abstract

We study dynamic multilateral markets, in which players' payoffs result from coalitional bargaining. We establish payoff uniqueness of stationary equilibria and the emergence of endogenous cooperation structures when traders experience some degree of (heterogeneous) bargaining frictions. When we focus on market games with different player types, we derive, under mild conditions, an explicit formula for each type's equilibrium payoff as market frictions vanish. We further apply this methodology to the analysis of labor markets. From our general results, we can determine the endogenous composition of the equilibrium firm and the remuneration scheme.

Keywords: multilateral bargaining, dynamic markets, labor markets

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1. Introduction

In many economic situations, the surplus from trade depends on the actions and preferences of multiple agents. This is certainly the case in inter-connected markets like supply chains, markets where intermediaries play a prominent role, for example financial or real estates markets and labor markets.

Our aim in developing a framework for the study of multilateral markets is to capture a competitive environment in which strategic behavior by players leads to the formation of prices. To this end, we extend the seminal model by Rubinstein and Wolinsky (1985) of a strategic buyer-seller market to multilateral markets. We also follow these authors in employing the notion of stationary market equilibrium for determining equilibrium prices. We establish the payoff uniqueness of the stationary equilibria in all games, in which markets are not perfectly frictionless (Theorem 1). [1]

To analyze in greater detail the properties of equilibrium payoffs, we extend our analysis to market games with heterogeneous types of players that may be treated as type-specific factors of production. Here, we focus on the equilibrium behavior of players when the bargaining frictions vanish. In this environment, we show that the limit equilibrium leads to an optimal cooperation structure, in which players of the same type receive identical limit equilibrium payoffs and the latter exhaust the values of the agreeing coalitions. This result implies the endogenous emergence of unique limit equilibrium price for each factor of production. Notably in the limit market equilibrium prices do not depend on the underlying matching technology and they are anonymous. In particular, each factor of production is rewarded (approximately) its marginal contribution to the coalition where agreement is reached (Theorem 2, items (i)-(iv)). Unlike Gul (1989) and Hart and Mas-Collel (1996), we find that the limit market equilibria of our market game are unrelated to the Shapley value of the corresponding static transferrable utility game. Finally, last of our general results (Theorem 3) fully characterize players' payoffs in the limit partitioning equilibria. In such

[1] Chatterjee, Dutta, Ray and Sengupta (1993) argue that invoking stationarity enhances the predictive power of the equilibrium analysis. The multiplicity of perfect equilibria in the extension of Rubinstein's (1982) model to multilateral bargaining, on the other hand, has been first pointed out by Shaked and reported by Sutton (1986).

an equilibrium, each player type cooperates in at most one coalition type, i.e., the set of player types is partitioned into subsets that cooperate in non-intersecting coalitions.

With its focus on dynamic multilateral bargaining, this paper contributes to two growing branches of literature: that on dynamic markets (for example, in the context of seller-buyer markets see Rubinstein and Wolinsky (1985), Rubinstein and Wolinsky (1990), and, more recently Duffie, Gâarleanu and Pedersen (2005), Manea (2011) and **? you may add your paper with Fernando here, too**); and that on coalitional bargaining (e.g. Chatterjee et al. (1993), Hart and Mas-Collel (1996), Chatterjee and Sabourian (2000), and more recently Okada (2011)). Based on its bargaining procedure, our work is closest in nature to Manea (2011). In fact, our Theorem 1 can be seen as a generalization of Manea's result on the payoff-uniqueness when the matched bilateral coalitions are network links with normalized values. **Add how our proof differs from that of Manea** Our work differs from other studies on multilateral markets in the bargaining procedure and matching mechanism. Several other works study multilateral bargaining, however, they do not consider a dynamic market. Some of these focus on characteristic function form games. These authors either aim at supporting cooperative allocations as equilibrium outcomes (cf. Gul (1989), Yan (2003)) or focus on the efficiency of the bargaining outcomes (cf. Chatterjee et al. (1993) and Okada (1996)). In a related contribution, Krishna and Serrano (1996) allow for "partial agreement" where the agreeing agents leave the bargaining procedure with their agreed shares, while the proposer and the disagreeing agents proceed to the next stage of bargaining. While the assumption of "partial agreement" may be valid in certain contexts, e.g., division of an estate, it is less applicable to others such as production with complementary inputs. Chatterjee and Sabourian (2000), instead, study unanimous agreement of all parties in the multi-person ultimatum game. They, however, assume that bargaining continues until agreement is reached. As our focus is on anonymous dynamic markets, the assumption that the bargaining coalition dissolves in case of disagreement seems more plausible. Finally, Okada (2011) studies coalitional bargaining in the context of coalition formation where a proposal is a pair of a coalition of players and a payoff vector.

A commonly studied example of multilateral interaction is the labor market with heterogenous workers. We choose to cast our analysis within this framework as this allows us to illustrate our theoretical findings and to transparently contrast them with existing results. In the labor market setup we develop, the limit equilibrium in the pertinent game turns out to be partitioning, hence, we can determine the endogenous firm structure (i.e., the composition of agreeing coalitions) and the remuneration scheme as functions of fundamentals. We find, in particular, that wages of different worker types reflect in a clear-cut manner the supply and demand forces in the market, adjusted by the type-specific bargaining power. We also find that each worker's share of the firm's value equals her marginal contribution to the firm. In essence, in our strategic competitive environment we recover the equality between wages and the marginal product of labor that characterizes the the neoclassical analysis. Unlike in the neoclassical case, however, in our framework factors of production may remain unemployed in equilibrium.

Our findings differ from those of other authors whose focus has been the study of the labor market, e.g., Stole and Zwiebel (1996a,b) and Okada (2011) and is similar in spirit to that of Westermark (2003). A crucial difference between this work and Westermark (2003), however, is that while we study multilateral bargaining, wages are determined by a sequence of bilateral bargaining events in his model. In particular, we find that the parity between the neoclassical wage and limit payoffs in this model coexists with some stylized facts shown in the search-matching literature[2] such as a positive level of equilibrium unemployment even when bargaining frictions vanish.[3]

The rest of the paper is organized as follows: In Section 2 we describe our theoretical framework and in Section 3 we state our main existence result. In Section 4, we specialize our discussion of market equilibria by introducing heterogeneous player types. In this context, we develop the notions of limit market equilibrium and of partitioning limit market equilibrium and we discuss their

[2] Delacroix (2003) and Yashiv (2007), for example, discuss related search-theoretic models of labor markets.

[3] Neither Stole and Zwiebel (1996a,b) or Westermark (2003) find a positive level of equilibrium unemployment in their models. Okada (2011), on the other hand, finds a possibility for a positive unemployment only if players have some degree of impatience.

characterization. We employ the theoretical results to the study of the labor market with homogeneous and heterogeneous labor force in Section 5. Section 6 provides some concluding remarks.

2. Market Interaction

We consider a market with a finite set $\mathcal{N}$ of replica agents that operates over an infinite number of discrete dates. As any agent $i \in \mathcal{N}$ that leaves the market is instantly replaced by its replica, the set $\mathcal{N}$ remains in the market throughout the entire history of the game. We interpret this market as embedded in a bigger economy. At any given time, firms and individuals arrive in the market looking for factors of production, business, or work opportunities. These agents search and bargain in the market and leave it when they have found satisfactory trades. However, they remain in the economy and may play an active role again when they need to replenish their resources. Think, for example, of a firm that needs to hire particular types of employees at unforeseeable dates. In this interpretation, participants in the economy return to the market periodically - possibly in different roles - but do not anticipate these events strategically. Our model describes, then, a steady state, when each agent that (temporarily) departs from the market is replaced by another that offers/demands the same input.

In our model, each date starts with a matching stage, in which non-intersecting coalitions of players (subsets of $\mathcal{N}$) are randomly matched. The stationary (exogenous) matching procedure implies, then, the probability π_S of selecting the coalition $S \subseteq \mathcal{N}$ that is constant throughout the game. When matched, the members of S can produce a surplus $v(S) \geq 0$ by employing their player-specific inputs. The production of $v(S)$ can only take place if all members of S agree on the terms of trade. The latter are negotiated either according to the Nash Bargaining Solution (NBS) or one of the players is chosen as proposer in an ultimatum bargaining game.[4] In the latter case, proposer $i \in S$ makes sequentially (in any order) offers to the players in $S \backslash \{i\}$. Like similar models that adopt the alternating offer mechanism of Rubinstein (1982), cf. Chatterjee

[4]As all coalitional members need to agree for a proposal to be carried out, the proposer makes an offer of what Huang (2002) refers to as *conditional* nature.

et al. (1993), we assume that the proposer is fully committed to the offers, once they are made. If an offer is rejected, coalition S dissolves immediately, i.e., without any further offers being made, and the same population of players proceeds to the next date. Otherwise, there is an agreement, in which case the value $v(S)$ is created, agents in $S\backslash\{i\}$ receive their agreed shares and i obtains the residual surplus. Then, the members of S leave the market and are replaced by their replicas with the same endowments before the game moves to the next date. Importantly, all new agents are treated by the matching procedure in the same way as the ones who left. In particular, the set of newcomers that have replaced the members of an agreeing coalition S, will be selected with probability π_S in all subsequent periods, in which they stay in the market.[5]

We parametrize the (absolute) bargaining power of player $i \in \mathcal{N}$ by α_i, which is assumed to be strictly positive and the same for all replicas of i. Let the set of all coalitions $S \subseteq \mathcal{N}$ containing player $i \in \mathcal{N}$ be denoted by S_i. Then, the probability that player i is proposer in $S \in S_i$ is given by $\alpha_i/\alpha(S)$, where $\alpha(S) := \sum_{k \in S} \alpha_k$, for all $i \in S$. In the context of Nash Bargaining, $\alpha_i/\alpha(S)$ is the (relative) bargaining power of i in coalition S. Finally, we assume that each replica of player i discounts future dates with the same factor $\delta_i \in (0, 1)$. We collect the discount rates in the vector $\delta = (\delta_1, \ldots, \delta_N)$. The fact that market participants face distinct discount factors may be due to differences in the outside options for their factors of production or in capital funding opportunities. Some of our results hold in the limit when δ_i approaches one for all $i \in \mathcal{N}$. We shall interpret this situation as vanishing market frictions.

In the next section, we explain how the threat points (or minimum acceptance levels) are determined endogenously in a stationary market equilibrium.

3. Stationary Equilibria

The market interaction described in the previous section defines a game with complete but imperfect information as we assume that players do not observe the terms of agreements they are not

[5] Similar assumptions are made by Manea (2011). Alternatively, given market fundamentals, we can focus on steady states in which, by definition, this condition is also satisfied.

part of.[6] One way to analyze this game is to follow, e.g., Rubinstein and Wolinsky (1985) - RW 1985 hereafter - and define its extensive form. This definition specifies histories and strategies for all players. In particular, for any agent, the history at a certain stage of the game is the sequence of observations made by her up to that stage. A strategy for any agent is, then, a sequence of decision rules conditional on all histories that dictates player's moves, i.e., the offers she makes as a proposer and her acceptance/rejection responses to offers made to her. RW 1985 focus on stationary strategies, i.e., strategies that prescribe a history-independent bargaining behavior towards the partners with whom an agent is matched. They define market equilibrium (ME thereafter) as a stationary strategy profile such that no agent can improve by changing her action after any possible history.

In this work, we consider only MEa and, therefore, we omit the formalization of histories and history-dependent strategies.[7] At any date t, a stationary strategy of proposer $i \in S$ in a matched coalition S will depend exclusively on the identities of responders in $S \backslash \{i\}$. Responder $j \in S \backslash \{i\}$, on the other hand, will condition her stationary strategy on the coalition S, the identity of the proposer $i \in S$ and on the offer made to her. It follows that for stationary strategy profile, matching environment and bargaining powers, each replica of player $k \in \mathcal{N}$ expects the same payoff x_k at the beginning of any date. We exploit this fact to characterize MEa by deriving conditions for ME payoffs $\{x_k\}_{k \in \mathcal{N}}$.

Specifically, if $j \in S \backslash \{i\}$ is the last responder in the matched coalition S, she will accept in equilibrium any offer larger than $\delta_j x_j$ and rejects any offer smaller than $\delta_j x_j$. If the last but one responder $k \in S \backslash \{i\}$ anticipates the acceptance by j, she will accept any offer larger than $\delta_k x_k$ and rejects any offer smaller than $\delta_k x_k$. This argument propagates to all responders in $S \backslash \{i\}$. If we define the N-dimensional vector $x_S := (x_i I_{i \in S})_{i=1,\dots,N}$, where $I_c = 1$ if c is true and $I_c = 0$

[6]This assumption is innocuous for our results that focus on stationary equilibria.

[7]Such formalization would be a straightforward generalization of that in Manea (2011) to coalitions with more than two members.

otherwise, then the proposer $i \in S$ will make only acceptable offers if,

$$v(S) - \delta x_{S\setminus\{i\}} > \delta_i x_i, \text{ or equivalently, } v(S) > \delta x_S.$$

Otherwise, the player i, who obtains the residual surplus $v(S) - \delta x_{S\setminus\{i\}}$, would be better off by increasing offers infinitesimally to ensure all sequential acceptances. If $v(S) < \delta x_S$, on the other hand, the residual claim is less than proposer's continuation payoff $\delta_i x_i$ and i will make at least one unacceptable offer. This will lead to disagreement in S and to the respective continuation payoffs for all members of S. Finally, if $v(S) = \delta x_S$, players in S are indifferent between agreement and disagreement, which can result in a randomized equilibrium agreements in S. This analysis of ME strategies is succinctly captured by the following system of N equations,

$$\begin{aligned} x_i &= \sum_{S\in\mathcal{S}_i} \pi_S \left(\frac{\alpha_i}{\alpha(S)} \max\{v(S) - \delta x_{S\setminus\{i\}}, \delta_i x_i\} + \frac{\alpha(S\setminus\{i\})}{\alpha(S)} \delta_i x_i \right) + (1 - \sum_{S\in\mathcal{S}_i} \pi_S)\delta_i x_i \\ &= \delta_i x_i + \sum_{S\in\mathcal{S}_i} \pi_S \frac{\alpha_i}{\alpha(S)} \max\{v(S) - \delta x_S, 0\}, \quad \forall i \in \mathcal{N}. \end{aligned} \tag{1}$$

A solution $x^\delta = (x_i^\delta)_{i\in\mathcal{N}}$ to system (1) yields expected ME payoffs. A unique solution, in particular, yields unique ME payoffs although these payoffs can be supported by multiple strategy profiles. Multiplicity occurs, for example, when responder j in the matched coalition S, for which it holds $v(S) = \delta x_S^\delta$, agrees with any probability to the offer $\delta_j x_j^\delta$. In what follows, (and somewhat ambiguously) we will often refer to a solution to (1) as ME.

We will say that S is *active* in ME x^δ if S agrees in this equilibrium with probability one. Further, we will say that player i *cooperates* in ME x^δ if $i \in S$ and S is active in x^δ. Note that re-writing (1) as,

$$x_i = (1 - \textstyle\sum_{S\in\mathcal{S}_i} \pi_S)\delta_i x_i + \textstyle\sum_{S\in\mathcal{S}_i} \pi_S \max\{\delta_i x_i + \dfrac{\alpha_i}{\alpha(S)}(v(S) - \delta x_S), \delta_i x_i\}, \tag{2}$$

makes clear that the expected payoff x_i results also when the outcome of each coalitional meeting is prescribed by the NBS, where player i's bargaining power in coalition S is given by $\alpha_i/\alpha(S)$ and the (endogenous) threat points are $(\delta_i x_i)_{i\in S}$.

We can interpret the payoff x_i as the price that player i expects for her input in a ME. This price would be different if agents entered the market with a stock of inputs. In such a case, the disagreement payoffs should internalize the utility loss due to delayed cooperations. Although we allow each replica agent to trade only once, our model can be easily extended in this direction.

The following Example 1 illustrates equilibrium payoffs in several stylized market settings.

Example 1. We consider a market with three players, $\mathcal{N} = \{1, 2, 3\}$, and discuss several cases of value functions that capture various standard market environments. To simplify our discussion and notation, in each case we only list the values of the productive coalitions and take $v(S) = \pi_S = 0$ for all unproductive coalitions $S \subseteq \mathcal{N}$. In addition, we will assume that all productive coalitions are matched with equal probabilities and that the sum of these probabilities is one.

Bilateral Bargaining First, we consider the bilateral game with $v(\{1, 2\}) = 1$. Then, the unique solution to the system (1) yields the ME payoffs,

$$x_1^\delta = \frac{\alpha_1(1-\delta_2)}{\alpha_1(1-\delta_2)+\alpha_2(1-\delta_1)}, \quad x_2^\delta = \frac{\alpha_2(1-\delta_1)}{\alpha_1(1-\delta_2)+\alpha_2(1-\delta_1)}, \tag{3}$$

and zero to the unproductive player 3.

For the sake of simplicity, we assume in the following examples equal discount factors, $\delta_i = d$, and equal bargaining powers, $\alpha_i = 1/3$. When we let $v(\{1, 2\}) = v(\{1, 3\}) = 1$, we obtain a simple two-sided market in which a single buyer (player 1) bargains with two identical sellers (players 2 and 3) over the price of one unit of a homogenous good. The unique ME solution here,

$$x_1^d = \frac{2}{4-d}, \quad x_2^d = x_3^d = \frac{1}{4-d},$$

illustrates the fact that the single buyer cannot extract the entire surplus from the competing sellers for any level of market frictions.

Generalizing the case above to a setting with two sellers of heterogenous goods, we set $v(\{1, 2\}) = 1$ and $v(\{1, 3\}) = a \geq 1$. Then, by taking the limit $d \to 1$ of the solution to (1), we obtain,

$$\begin{aligned} x_1^1 &= \frac{1+a}{3}, \quad x_2^1 = \frac{2-a}{3}, \quad x_3^1 = \frac{2a-1}{3}, \quad \text{if} \quad a \leq 2, \\ x_1^1 &= a/2, \quad x_2^1 = 0, \quad x_3^1 = a/2, \quad \text{if} \quad a > 2. \end{aligned}$$

Thus, both buyer-seller pairs trade when $a < 2$, while only coalition $\{1,3\}$ agrees when $a > 2$. When $a < 2$, the price of player 3's good increases in its value a and this is mirrored by the decrease of the price of the good of player 2.

Multilateral bargaining As in the benchmark bilateral example before, when only the three-player coalition is productive, $v(\{1,2,3\}) = 1$, the outcome of the market game is the even division of the surplus, i.e., $1/3$ for each player, irrespective of the discount factor d.

Alternatively, multilateral bargaining may be captured by $v(S) = 1$ for all S with at least two members. The unique solution to the system (1) for $d > 4/5$,[8]

$$x_i^d = \frac{1}{2(2-d)}, \quad i = 1,2,3,$$

implies an agreement by any matched pair of players and a disagreement in the grand coalition. It also shows that the ME payoffs can differ from the Shapley values $(1/3, 1/3, 1/3)$ of the corresponding static game and that they are not in the core of this game, which is empty.

Finally, we consider a stylized example of intermediation. We let $v(\{1,2\}) = v(\{1,2,3\}) = 1$ be a market with one buyer (player 1), one seller (player 2) and an (unproductive) intermediary (player 3). The solution to (1),

$$x_1^d = x_2^d = \frac{10-9d}{2(12-12d+d)}, \quad x_3^d = \frac{4(1-d)}{2(12-12d+d)}, \tag{4}$$

shows that the unique ME payoff to the buyer (seller) increases with d, reaching $1/2$ in the limit $d \to 1$. The unique equilibrium payoff for the intermediary, on the other hand, is decreasing with d, reaching 0 when $d \to 1$. As expected, the intermediary's profit vanishes as the market becomes frictionless.

These examples of games with three players illustrate the richness of the framework and its

[8]For $d < 4/5$ the solution $x_i^d = \frac{4}{3(4-d)}$ to (1) implies an agreement in all matched coalitions. For $d = 4/5$, we have that $3d\frac{4}{3(4-d)} = 3d\frac{1}{2(2-d)} = 1$ and an agreement with any probability is rational in the grand coalition.

broad applications in various fields of economics. In particular, different values of coalitions can reflect their varying productivities, possibly due to their type composition (which we explore in the next section). On the other hand, the matching procedure can reflect the strength of (multilateral) connections among agents. For example, it can simulate a weighted network, in which only connected pairs of agents are matched with positive probabilities. As a conclusion of this section, the next proposition shows the uniqueness of market equilibria.

Theorem 1. There exists a unique ME payoff profile in any market game.

Proof: All proofs are relegated to the Appendix.

4. Limit Market Equilibria and Multilateral Coalitions

In this section, we focus on a class of dynamic market games, where players of distinct types interact. Specifically, we consider a partitioning of the set of agents $\mathcal{N}$ into T types, i.e., $\mathcal{N} = \cup_{t=1,...,T} \mathcal{N}_t$ and $\mathcal{N}_s \cap \mathcal{N}_t = \varnothing$ for all $s, t = 1, \ldots, T$ with $s \neq t$. For all types $t = 1, \ldots, t$, we denote the cardinality of the set $\mathcal{N}_t$ by N_t. A multilateral coalition (MC) $S \subseteq \mathcal{N}$ consists of $\sum_{t=1,...,T} n_t$ players, where $n_t = |S \cap \mathcal{N}_t|$ denotes the number of t-type players in this coalition. For any MC $S \subseteq \mathcal{N}$, we use the operator $T(S)$ to obtain the type profile of its members. Formally, given $S \subseteq \mathcal{N}$, its type $T(S)$ is defined as an ordered vector of type multiplicities, i.e., $T(S) = (n_1, ..., n_T)$ where $n_t = |S \cap \mathcal{N}_t|$ for $t = 1, ..., T$. For example, for the grand coalition $T(\mathcal{N}) = (N_1, ..., N_T)$, while for a singleton coalition $\{i\}$ with $i \in \mathcal{N}_t$, we obtain $T(\{i\}) = \mathbf{e}_t$, where $\mathbf{e}_t$ is a vector in the canonical basis of the Euclidean space that points in direction t.

We make the natural assumption that coalitions of the same type have the same productivity, i.e., $v(S) = v(S')$ when $T(S) = T(S')$. Therefore, we can use the shorthand notation $v(T(S))$ for the productivity of any coalition of type $T(S)$. Furthermore, we assume that each productive MC S has a positive probability of meeting, $\pi_S > 0$ if $v(S) > 0$. Finally, a symmetric treatment of players of the same type requires that such players have equal absolute bargaining powers and discount factors, $\alpha_i = \alpha_j$ and $\delta_i = \delta_j$ if $T(\{i\}) = T(\{j\})$. Similar to Manea (2011) and Chatterjee et al. (1993), we study market outcomes when bargaining frictions vanish. In order to avoid technicalities, we will assume equal discount factors $\delta_t = d$ for all types t, and derive our results when d converges to one. Our aims are to characterize agreements in coalitions of different types and derive explicit expressions for the players' limit equilibrium payoffs. Formally, a limit market equilibrium is defined below.

Definition 1. A ME x^1, where $x^1 := \lim_{d\to 1} x^d$ and x^d is a solution to (1) will be called a limit ME (LME).

Note that LME is well-defined and, importantly, it implies a unique set of active coalitions $\{S \subseteq \mathcal{N} : v(S) > dx^d(S)\}$ for all $d \in (\underline{d}, 1)$ and a sufficiently high threshold $\underline{d} < 1$. This follows from the uniqueness of the solution x^d, its continuity in d, and from a straightforward generalization of Theorem 2 in Manea (2011).[9]

The following result establishes payoff-equivalence between players of the same type, the value exhaustion for each active MC, the bounds on the LME payoffs and the optimality of the LME. In order to show the last property, we define a payoff profile y^P feasible with respect to a set of active coalitions $P = \{S_1, \dots, S_p\}$ if $y^P(S_k) \leq v(S_k)$ for all active coalitions $S_k \in P$ and $y_i^P = 0$ for all agents i who are not members of an active coalition. Using the definition of a feasible payoff, we say that a feasible payoff profile, x^{P*}, is *optimal* if $x^{P*}(N) \geq y^P(N)$ for any profile of active coalitions P and any feasible payoff profile y^P with respect to P. We thus define an outcome to be optimal if it maximizes the joint payoff of all agents where no transfers across coalitions are permitted. .

Theorem 2. Consider a market game and a LME x^1 of this game. Then

(i) all players of type $t = 1, \dots, T$, receive the same payoff x_t^1;

[9] In Manea (2011), all coalitions (links) contain two players and their surplus is normalized to one. Manea's Theorem 2 proves, then, that there is a $\underline{d} < 1$ such that for all $d \in (\underline{d}, 1)$, the set of active coalitions remains the same.

(ii) for each coalition of type $\mathbf{n} = (n_1, \ldots, n_T)$, it holds that

$$\sum\nolimits_{s=1}^{T} n_s x_s^1 \geq v(\mathbf{n}),$$

where equality holds for each active coalition.

(iii) a player of type t that cooperates in an LME in coalitions of type $\mathbf{n}$ receives (approximately) her marginal contribution to this type of coalitions,

$$v(\mathbf{n} + \mathbf{e}_t) - v(\mathbf{n}) \leq x_t^1 \leq v(\mathbf{n}) - v(\mathbf{n} - \mathbf{e}_t), \tag{5}$$

whenever $v(\mathbf{n} + \mathbf{e}_t)$ and $v(\mathbf{n} - \mathbf{e}_t)$ are well-defined.[10]

(iv) LME x^1 is optimal, i.e., for any feasible payoff profile $\{S^k, y_t^k\}_{t=1,\ldots,T}^{k\in\mathcal{K}}$ and each coalition S^k active in this profile, $x^1(S^k) \geq y^k(S^k)$.

In order to refine the results in the last theorem, we define a subset of LMEa that we shall call partitioning equilibria. Intuitively, an LME is partitioning if each player type cooperates in at most one coalition type, i.e., the set of player types is partitioned into subsets that cooperate in non-intersecting coalitions. Formally,

Definition 2. A LME x^1 is partitioning (PLME) if,

$$\forall S, S' \subseteq \mathcal{N} : S \cap S' \neq \varnothing, \quad x^1(S) \leq v(S) \quad \& \quad x^1(S') \leq v(S') \Rightarrow T(S) = T(S'). \tag{6}$$

In a PLME, intersecting coalitions will rationally cooperate only if these coalitions are of the same type. Partitioning equilibria will arise naturally in many games, e.g., when only one type of coalition is productive as in the homogenous buyer-seller market discussed in Example 1. However, one can easily construct examples, where the unique LME is not partitioning. For instance, in the intermediation game, discussed last in Example 1, the members of the buyer-seller coalition $S = \{1, 2\}$ obtain the joint limit payoff of one, which is the same as the joint limit payoff of

[10]The unit vector $\mathbf{e}_t$ belongs to the canonical basis of the Euclidean space and points in direction t.

the members of coalition $S' = \{1, 2, 3\}$. As $x^1(S) = v(S) = 1$ and $v(S') = v(S') = 1$ but $T(S) = (1, 1, 0) \neq (1, 1, 1) = T(S')$, this LME is not partitioning.

An appealing property of partitioning equilibria, as described in the next theorem, is that when such equilibria exist in a market game the PLME payoffs can be fully characterized as functions of market fundamentals.

Theorem 3. In the PLME x^1, all players of type $t = 1, \ldots, T$, who cooperate in coalitions of type $\mathbf{n}$, receive

$$x_t^1 = v(\mathbf{n}) \frac{n_t \alpha_t / N_t}{\sum_{s=1}^{T} (n_s^2 \alpha_s / N_s)}. \tag{7}$$

A notable common feature of the results summarized in Theorems 2 and 3 is that they are independent of the matching procedure. An agent's equilibrium payoff depends instead on the characteristic value function of the feasible coalitions of which she is a member. We explore the significance of these results in a stylized example in the next section.

5. Dynamic Labor Market

A discussion of a labor market where workers (with various skills) bargain over a remuneration scheme, taking into account firm's value and their own (endogenous) outside options allows us to illustrate our main findings so far. We choose the labor market as an application as it allows us to contrast our results to those derived in the analysis of neoclassical markets. We start off by discussing a labor market with homogeneous workers, for which we derive the equilibrium wage and the firm size. Subsequently, we generalize our results to a heterogeneous input market.

5.1. Homogeneous Labor Market

At each date, the labor market consists of N_e homogeneous entrepreneurs where a generic entrepreneur is indexed i, and N_w homogenous workers, where a generic worker is indexed by w. We will denote the set of entrepreneurs and the one of workers by $\mathcal{N}_e$ and $\mathcal{N}_w$, respectively. Hence, the set of all players $\mathcal{N} = \mathcal{N}_e \cup \mathcal{N}_w$. Each entrepreneur (worker) has the same same bargaining power α_i (α_w) and discount factor δ_i (δ_w). The productivity of a coalition with $n \leq N_w$ workers and an entrepreneur (a productive coalition) is given by a strictly concave production function $F(n) : \mathbb{N} \mapsto \mathbb{R}_+$, i.e., $v(S) = F(n)$ for all $S \subseteq \mathcal{N}$ such that $|S \cap \mathcal{N}_e| = 1$ and $|S \cap \mathcal{N}_w| = n$. In particular, production is impossible without workers, $F(0) = 0$, or without an entrepreneur, $v(S) = 0$ for all $S \subseteq \mathcal{N}_w$.

We can compute the ME payoffs for an entrepreneur and a representative worker, x_i^δ and x_w^δ, respectively, from (1) when only coalitions with n workers and one entrepreneur are matched. For symmetric matching probabilities, system (1) is reduced to two equations,

$$\begin{aligned} x_i &= \delta_i x_i + \frac{1}{N_e}\frac{\alpha_i}{\alpha(n)}\max(F(n) - \delta_i x_i - n\delta_w x_w, 0), \\ x_w &= \delta_w x_w + \frac{\binom{N-1}{n-1}}{\binom{N}{n}}\frac{\alpha_w}{\alpha(n)}\max(F(n) - \delta_i x_i - n\delta_w x_w, 0), \end{aligned} \tag{8}$$

where $\alpha(n) := \alpha_i + n\alpha_w$. It can be shown that the solution (x_i^δ, x_w^δ) to the latter system satisfies,

$$\frac{x_i^\delta}{x_w^\delta} = \frac{\alpha_i N_w(1-\delta_w)}{\alpha_w n N_e(1-\delta_i)}, \tag{9}$$

Notably, the expression (9) depends in an intuitive way on the relative bargaining power and relative discount factor, and it is a decreasing function of the labor market tightness, nN_e/N_w, where n is the number of vacancies in a firm, N_e is the number of potential firms, and N_w measures total labor supply. In particular, when the market is tight (nN_e/N_w is low) an entrepreneur extracts a higher share of the value of the firm and the opposite holds, when there is relatively more vacancies (relatively larger demand as measured by firm size and number of firms) than the number of workers looking for a job.

From the solution (x_i^δ, x_w^δ) to (8), we can calculate the limit payoffs when $\delta_i = \delta_w = d$ approach one,

$$\begin{gathered} \lim_{d\to1} x_i^d = x_i^1 = \frac{\alpha_i F(n) N_w}{\alpha_i N_w + \alpha_w n^2 N_e}, \quad \lim_{d\to1} x_w^d = x_w^1 = \frac{\alpha_w F(n) n N_e}{\alpha_i N_w + \alpha_w n^2 N_e}, \\ \frac{x_i^1}{x_w^1} = \frac{\alpha_i N_w}{\alpha_w n N_e}, \quad x_i^1 + n x_w^1 = F(n). \end{gathered} \tag{10}$$

which do not depend on the matching probabilities and are a special case of the limit payoffs specified in item (ii) of Theorem 3. Moreover,

$$\partial x_w^1/\partial N_w < 0, \quad \partial x_i^1/\partial N_w > 0 \quad \text{and} \quad x_w^1/\partial N_e > 0, \quad \partial x_i^1/\partial N_e < 0.$$

The above derivatives are similar to the findings in search models, e.g., Shapiro and Stiglitz (1984): the tighter the labor market (i.e., the higher the labor supply N_w for a given number of vacancies nN_e a) the lower the worker's wage x_w^1. The reverse is expected to hold with respect to the entrepreneur's payoff. In addition, we find that the higher the competitive pressure on the demand side (i.e. the higher the number of competing firms for workers, N_e), the lower the entrepreneur's payoff and the higher the worker's wage. Furthermore, we can re-write Theorem 2(iii) as

$$F(n) - F(n-1) \geq x_w^1 \geq F(n+1) - F(n). \tag{11}$$

Essentially, the above expressions establish the parity between the limit payoff and the neoclassical wage. It entails that the outcome of our multilateral bargaining procedure results in each worker being paid approximately her marginal contribution to the firm value. Given the assumption of homogenous labor, we can interpret the marginal contribution as the marginal product of labor.

So far, our assumption that only coalitions with n workers are matched led to the equilibrium payoffs (10). In general, many productive coalitions may have a (random) opportunity to cooperate. We discuss this case for large labour markets in what follows. Formally, we shall consider the argument of the production function F as a continuous variable. Then, as we show in the more general environment of the next subsection, the limit ME is partitioning and a unique firm size and unique limit payoffs (10) emerge endogenously. Each player type in this PLME earns the limit payoff (7), as specified in Theorem 3.

To facilitate our discussion of how the equilibrium firm size is related to the market fundamentals, i.e., technology and input abundance, we will assume that the firm is endowed with a Cobb-Douglas production function, $F(z) = Az^{\gamma}$. In order to simplify the expressions, we assume, further, that the bargaining power of the entrepreneur in the PLME firm is the same as the collective bargaining power of the workers, $\alpha_i = \alpha_w n$.[11] Then the limit equilibrium payoffs (10) simplify to,

$$x_i^1 = \frac{F(n)N_w}{N_w + nN_e}, \quad x_w^1 = \frac{F(n)N_e}{N_w + nN_e}. \tag{12}$$

Recall that (11) indicates that the wage should approximate the product of the marginal worker. Hence, the limit equilibrium firm size must satisfy,[12]

$$x_w^1 = \frac{F(n)N_e}{N_w + nN_e} = F'(n) = \frac{\gamma}{n}F(n) \Rightarrow n = \frac{N_w\gamma}{N_e(1-\gamma)} \quad for \quad \gamma < \frac{N_e}{1+N_2}. \tag{13}$$

Notably, we obtain a full characterization of the PLME - the unique firm size, wages, and profit - without any reference to an exogenous reservation wage (the outside option of the worker). Notice that the equilibrium firm size is larger the greater the total labor supply N_w relative to the number of firms N_e. Furthermore, an increase in γ - i.e., a more productive technology - leads to a larger equilibrium firm size n, lower unemployment rate $(N_w - n)/N_w$ and a higher equilibrium profit and wage,

$$x_i^1 = \frac{N_w}{N_e}x_w^1, \quad x_w^1 = \frac{F(n)N_e}{N_w + nN_e} = A\left(\gamma\right)^\gamma\left(1-\gamma\right)^{1-\gamma}\left(\frac{N_e}{N_w}\right)^{1-\gamma},$$

as the expression on the r.h.s. of the last equality increases in γ when $n = N_w\gamma/N_e(1-\gamma) > 1$.

So far, we have discussed labor market equilibria with vanishing bargaining frictions. In markets with frictions, we can further replicate the stylized fact that players of the same type may obtain a different expected payoff. This is driven by the matching environment and the fact that players of the same type may reach an agreement in coalitions of distinct types in equilibrium.[13]

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[11] Without this assumption, we can establish qualitatively the same results but we will have to deal with more complicated expressions that depend on bargaining powers.

[12] Clearly, $\gamma > N_e/1 + N_e$ implies $n > N_w$. In this case, it can be shown that $n = N_w$ in the PLME.

[13] As an illustration, consider the following example. Let the set of workers be $\mathcal{N}_w = \{1, 2\}$ and the entrepreneur i. Let the coalition surplus be $v(\{i, 1, 2\}) = 2$, $v(\{i, 1\}) = v(i, 2) = 1$, and $v(S) = 0$ for any other $S \subseteq \{i, 2, 3\}$. Let $\pi_{\{i,1,2\}} = 1/5$, $\pi_{\{i,1\}} = 3/5$, and $\pi_{\{i,2\}} = 1/5$; and $\delta = 0.9$. Then, all productive coalitions agree in ME and $x_i = 0.5693$, $x_1 = 0.5052$ and $x_2 = 0.4705$.

5.2. Labor Market with Heterogeneous Inputs

This section generalizes the results derived for homogeneous labor markets. The labor market consists of T types of players of which type one is entrepreneurs and the remainder $T-1$ types are workers. Hence, at each date, there are N_1 entrepreneurs and N_t workers of type $t = 2, \ldots, T$. Again, we assume that only coalitions that contain exactly one entrepreneur and a positive number of workers are productive. The production function takes now the form $F(z_2, \ldots, z_T) : \mathbb{N}^{T-1} \mapsto \mathbb{R}_+$, which is strictly concave and where the argument z_t refers to the number of t-type workers employed by the firm. Alternatively, we can interpret F as the output of a hierarchy with T levels with exactly one entrepreneur at the top level levels and z_t employees at each level $t = 2, \ldots, T$. When $\delta_1 = \ldots = \delta_N = d$ approach one, part (ii) of Theorem 2 takes the form,

$$F(\mathbf{n}) - F(\mathbf{n} - \mathbf{e}_t) \geq x_t^1 \geq F(\mathbf{n} + \mathbf{e}_t) - F(\mathbf{n}), \quad \text{for all } t = 2, \ldots, T,$$

which generalizes our findings about the relationship between the LME payoffs and the neoclassical wage. The inequalities imply that for each type, a worker of that type receives (approximately) her marginal contribution to the firm value.

As in the previous section, we consider all arguments of F as continuous variables in the ensuing discussion.[14] Then, in order to show that the LME x^1 is partitioning, we consider - for the sake of contradiction - two different profiles, $\mathbf{n}$ and $\mathbf{m}$, that are active in this LME, i.e.,

$$F(\mathbf{n}) = x_1^1 + \sum\nolimits_{t=2}^{T} n_t x_t^1, \quad F(\mathbf{m}) = x_1^1 + \sum\nolimits_{t=2}^{T} m_t x_t^1,$$

by the Theorem 2(ii). We combine both equalities and obtain the desired contradiction,

$$F(\frac{\mathbf{n}+\mathbf{m}}{2}) > \frac{1}{2}F(\mathbf{m}) + \frac{1}{2}F(\mathbf{n}) = x_1^1 + \sum\nolimits_{t=2}^{T} \frac{n_t + m_t}{2} x_t^1 \geq F(\frac{\mathbf{n}+\mathbf{m}}{2}),$$

where the first inequality follows from the strict concavity of F and the second from Theorem 2(ii), when applied to worker coalitions of type $(\mathbf{n}+\mathbf{m})/2$. Therefore, we can use Theorem (2)

[14]The results for continuous arguments of F can be approximated by increasing the (integer) number N_t of workers of each type t and defining $F(\mathbf{n}) := f(\mathbf{n}/\mathbf{N})$ for a concave function f.

to compute the PLME payoffs and the relative limit wages of different types of workers (different levels in a hierarchy),

$$\frac{x_t^1}{x_s^1} = \frac{n_t \alpha_t N_s}{n_s \alpha_s N_t}, \quad s,t = 1,\dots,T. \tag{14}$$

The latter relationship reveals that the relative scarcity of each type in the market influences the relative wages in the expected way: a higher supply N_t of type t workers depresses their relative individual wages ceteris paribus. On the other hand, the total number n_t of workers of type t, employed in an equilibrium firm, has a positive impact on their relative wages. This is not surprising, when we observe that n_t represents the demand for this type by the equilibrium firm. Therefore, (14) combines in a clear-cut manner the supply and demand forces in the market, adjusted by the type-specific bargaining power.

Next, we consider a Cobb-Douglas production function with $T-1$ worker types,

$$F(\mathbf{n}) = A \prod\nolimits_{t=2}^{T} n_t^{\gamma_t}, \quad \text{with } \mathbf{n} = (n_2,\dots,n_T), \quad \gamma_t > 0, \quad \sum\nolimits_{t=2}^{T} \gamma_t < 1.$$

Recall that entrepreneurs are denoted as type 1 players. Hence $n_1 = 1$ in any active coalition. It will turn out convenient to denote $\gamma_1 := 1 - \sum_{t=2}^{T} \gamma_t$.

For the sake of simplicity, we assume again equal bargaining power for each group in the equilibrium firm,

$$n_s \alpha_s = n_t \alpha_t, \quad \text{for all } t,s = 1,...,T. \tag{15}$$

Similar to our analysis of the equilibrium firm size in the homogeneous labor market, we can derive the unique equilibrium firm composition $\mathbf{n}$ by setting the limit PLME payoff x_t^1 equal to the marginal contribution of this type,

$$x_t^1 = \frac{\partial F(\mathbf{n})}{\partial n_t} = \frac{\gamma_t}{n_t} F(\mathbf{n}) \text{ for all } t = 2,\dots,T. \tag{16}$$

As the PLME payoffs (7) exhaust the surplus $F(\mathbf{n})$,

$$F(\mathbf{n}) = x_1^1 + \sum\nolimits_{t=2}^{T} n_t x_t^1 = x_1^1 + \sum\nolimits_{t=2}^{T} \gamma_t F(\mathbf{n}) \Rightarrow x_1^1 = \gamma_1 F(\mathbf{n}).$$

Using (16), the limit payoffs (7), and the equal bargaining power assumption, (15), we can derive the equilibrium quantity of factor inputs and equilibrium factor payoffs,

$$n_t = \frac{\gamma_t}{\gamma_1}\frac{N_t}{N_1}, \quad x_t^1 = \frac{A}{N_t}\prod\nolimits_{t=2}^{T}(\gamma_s N_s)^{\gamma_s}, \quad \text{for } t = 2,\ldots,T.$$

Here we assume an interior solution $1 < n_t < N_t$ (or, equivalently, $\gamma_t < \gamma_1$) for each type $t = 2,\ldots,T$. In an interior PLME, an increase in productivity of type s leads to higher employment levels and higher payoffs for all types of workers and the entrepreneur,

$$\frac{\partial n_t}{\partial \gamma_s} > 0 \quad \text{and} \quad \frac{\partial x_t^1}{\partial \gamma_s} > 0 \quad \text{and} \quad \frac{\partial x_i^1}{\partial \gamma_s} > 0 \quad \text{for all } t, s = 2,\ldots,T.$$

On the other hand, increasing the supply of type s decreases the payoff of this type, while it increases the demand for it and the payoffs to all other players,

$$\frac{\partial x_s^1}{\partial N_s} < 0, \quad \frac{\partial n_s}{\partial N_s} > 0, \quad \frac{\partial x_t^1}{\partial N_s} > 0, \quad \frac{\partial x_i^1}{\partial N_s} > 0, \quad t, s = 2, ..., T, t \neq s.$$

Notice that in the PLMEa discussed above not all coalitions which are matched reach an agreement. Thus some workers choose to remain unemployed in anticipation of higher wages and some firms choose to keep vacancies unfilled in the anticipation of higher profits. This insight complies with the stylized facts found in the search-theoretic literature, cf. Yashiv (2007). We derive these results, however, in an environment where market frictions, as captured by the matching probabilities, do not matter for the equilibrium behavior. Moreover, in our framework positive equilibrium level of unemployment co-exists with the neoclassical competitive equilibrium rule that is rewarding all inputs their marginal product.

6. Concluding remarks and extensions

This work develops a price-setting mechanism that makes explicit the role of strategic behavior in the context of dynamic multilateral markets. We have shown the existence and payoff uniqueness of stationary equilibria in any multilateral market game, in which players exhibit at least some degree of impatience. When we apply this framework to the analysis of the labor market, our procedure results in equilibrium prices that equal the respective marginal product of the factors of production. In this respect, our model provides an alternative, strategy-based microeconomic explanation of this feature of neoclassical markets. Our analysis also allows for an endogenous determination of the composition of a firm with heterogeneous inputs without any reference to the outside options of players. In this setting, we find support for a number of stylized facts discussed in the search-theoretic models, such as the existence of voluntary unemployment and unfilled vacancies, the dependence of wages on the relative scarcity of skill types, and that workers of homogeneous skill types may obtain different equilibrium payoffs.

Further extensions of the current study to institutional design can be done in the context of multilateral coalition bargaining. Similar to Stole and Zwiebel (1996a) one can investigate how the interplay of factors' productivity, factors' abundance, and bargaining power shape up the equilibrium hierarchical depth and width of an organization.

We also want to stress the straightforward applicability of our results to markets that exhibit multilateral structure of interactions, such as the housing market, credit-card market or retailing. For example, an application of (7) to a market game with T types of players, where only coalitions composed of one player of each type are productive, results in unique PLME payoffs.[15] Similar intuition to the analysis of two-sided markets can be derived here, too. A player's payoff is increasing with her bargaining power and with the relative weighted scarcity of players of her own type with weights equal to the bargaining power of each type. Moreover, the equilibrium payoff of each player is positively related to the number of players of the opposite type and negatively related the number of players of her own type.

[15] In this PLME, $x_t^1 = \frac{\alpha_t/N_t}{\sum_s(\alpha_s/N_s)}, \quad \frac{x_t^1}{x_s^1} = \frac{\alpha_t N_s}{\alpha_s N_t} \quad$ for $t, s = 1, \ldots, T$.

7. Appendix

Lemma 1. *For any $n \times n$ matrix A and diagonal $n \times n$ matrices D_α and D_δ with positive determinants, let $\Sigma = D_\alpha^{1/2} D_\delta^{1/2} A D_\alpha^{1/2} D_\delta^{1/2}$. Then,*

$$1) \quad \lambda(D_\alpha A D_\delta) = \lambda(\Sigma), \quad 2) \quad \lambda(D_\delta - D_\alpha A D_\delta) = \lambda(D_\delta - \Sigma),$$

where $\lambda(M)$ denotes an eigenvalue of the matrix M.

PROOF. Proof: Let $\widetilde{z} = (D_\alpha^{1/2} D_\delta^{-1/2})z$.

$$1) \quad \Sigma = D_\alpha^{1/2} D_\delta^{1/2} A D_\alpha^{1/2} D_\delta^{1/2} = (D_\alpha^{-1/2} D_\delta^{1/2})(D_\alpha A D_\delta)(D_\alpha^{1/2} D_\delta^{-1/2}) \Rightarrow$$
$$(D_\alpha^{-1/2} D_\delta^{1/2})(D_\alpha A D_\delta)(D_\alpha^{1/2} D_\delta^{-1/2})z = \lambda z \Rightarrow (D_\alpha A D_\delta)\widetilde{z} = \lambda \widetilde{z}.$$

$$2) \quad D_\delta - \Sigma = D_\delta - (D_\alpha^{-1/2} D_\delta^{1/2})(D_\alpha A D_\delta)(D_\alpha^{1/2} D_\delta^{-1/2}) \Rightarrow$$
$$D_\delta z - (D_\alpha^{-1/2} D_\delta^{1/2})(D_\alpha A D_\delta)(D_\alpha^{1/2} D_\delta^{-1/2})z = \lambda z \Rightarrow D_\delta \widetilde{z} - (D_\alpha A D_\delta)\widetilde{z} = \lambda \widetilde{z}. \blacksquare$$

Lemma 2. *The system*

$$x_i = \delta_i x_i + \sum\nolimits_{S \in \mathcal{S}_i} \theta_S \pi_S \frac{\alpha_i}{\alpha(S)} (v(S) - \delta x_S), \quad \forall i \in \mathcal{N}, \tag{17}$$

has a unique solution $x(\theta)$ for any (fixed) vectors of agreement probabilities $\theta = \{\theta_S\}_{S \subseteq \mathcal{N}}$.

PROOF. Let $\mathbf{I}_c = 1$ if c is true and $\mathbf{I}_c = 0$ otherwise, $x_S := (x_i \mathbf{I}_{i \in S})_{i=1}^N \in R^N$ and define f as,

$$f_i(x) = \delta_i x_i + \sum\nolimits_{S \in \mathcal{S}_i} \theta_S \pi_S \frac{\alpha_i}{\alpha(S)} (v(S) - \delta x_S) = \Phi x + \phi, \tag{18}$$
$$\Phi = (I - D_\alpha \Pi) D_\delta, \quad I, D_\alpha, D_\delta, \Pi \in R_+^{N \times N}, \phi \in R_+^N,$$
$$I = \mathbf{I}_{i=j},\ (D_\alpha)_{ij} = \mathbf{I}_{i=j}\alpha_i,\ (D_\delta)_{ij} = \mathbf{I}_{i=j}\delta_i,\ \phi_i = \alpha_i \sum\nolimits_{S \in \mathcal{S}_i} \gamma_S v(S),$$
$$\Pi_{ij} = \sum\nolimits_{S \in \mathcal{S}_i \cap \mathcal{S}_j} \gamma_S, \quad \gamma_S = \theta_S \pi_S / \alpha(S) \geq 0, \quad \forall i, j \in \mathcal{N},$$

Hence, D_α and D_δ are diagonal matrices and Π is a non-negative symmetric matrix with all real eigenvalues.

First, we prove that all eigenvalues of Φ lie in the interval $[-\widehat{\delta}, \widehat{\delta}]$, where $\widehat{\delta} = \max_i \delta_i < 1$. We note that the eigenvalues of $D_\alpha \Pi D_\delta$ are bounded from above by $\widehat{\delta}$,

$$\forall j \in \mathcal{N}, \quad \sum\nolimits_{i=1}^{N} (D_\alpha \Pi D_\delta)_{ij} = \sum\nolimits_{i=1}^{N} \alpha_i \delta_j \sum\nolimits_{S \in \mathcal{S}_i \cap \mathcal{S}_j} \gamma_S$$
$$\leq \delta_j \sum\nolimits_{i=1}^{N} \alpha_i \sum\nolimits_{S \in \mathcal{S}_i \cap \mathcal{S}_j} \frac{\pi_S}{\alpha(S)} = \delta_j \sum\nolimits_{S \in \mathcal{S}_j} \pi_S \leq \delta_j$$
$$\Rightarrow \lambda_{\max}(D_\alpha \Pi D_\delta) \leq ||D_\alpha \Pi D_\delta||_1 \leq \widehat{\delta}.$$

In order to show the lower bound on the eigenvalues of $D_\alpha \Pi D_\delta$, we note that Π is positive semidefinite,

$$\forall z \in R^N, \quad z^T \Pi z = \sum\nolimits_{S \subseteq \mathcal{N}} \gamma_S \left(\sum\nolimits_{i \in S} \sum\nolimits_{j \in S} z_i z_j \right) = \sum\nolimits_{S \subseteq \mathcal{N}} \gamma_S \left(\sum\nolimits_{i \in S} z_i \right)^2 \geq 0,$$

which implies that all eigenvalues of Π are nonnegative. As Π is symmetric, it can be diagonalized, i.e. $\Pi = P \Lambda P^T$, where the diagonal matrix Λ contains the nonnegative eigenvalues of Π. Let

$$\Sigma := D_\delta^{1/2} D_\alpha^{1/2} \Pi D_\alpha^{1/2} D_\delta^{1/2} = (D_\delta^{1/2} D_\alpha^{1/2} P) \Lambda (D_\delta^{1/2} D_\alpha^{1/2} P)^T.$$

By Sylvester's law of inertia, the number of negative eigenvalues is the same for the symmetric matrix Σ and for Λ. As the latter diagonal matrix has only nonnegative entries, the same must hold for the former. On the other hand, Σ has the same eigenvalues as $D_\alpha \Pi D_\delta$ by Lemma 1.1. Hence, we conclude that all eigenvalues of $D_\alpha \Pi D_\delta$ and of Σ lie between 0 and $\widehat{\delta}$.

Then, applying Weyl's inequality to the symmetric matrix $D_\delta - \Sigma$, we obtain the bounds,

$$\begin{aligned} \lambda_{\min}(D_\delta - \Sigma) \quad &\geq \quad \lambda_{\min}(D_\delta) + \lambda_{\min}(-\Sigma) > -\widehat{\delta} > -1, \\ \lambda_{\max}(D_\delta - \Sigma) \quad &\leq \quad \lambda_{\max}(D_\delta) + \lambda_{\max}(-\Sigma) \leq \widehat{\delta} < 1. \end{aligned}$$

By Lemma 1.2, $D_\delta - \Sigma$ and $\Phi = D_\delta - D_\alpha \Pi^x D_\delta$ have the same set of eigenvalues, which proves that all eigenvalues of Φ are less than one in modulus. By the Contraction Mapping Theorem and Lemma 2.1 in Bramoulle (2001), it follows that (17) has a unique solution $x(\theta) \in R^N$. ■

Lemma 3. *A ME x is continuous in $v(S)$ for any coalition $S \in \Theta = \{S \subseteq \mathcal{N} : \pi_S > 0\}$.*

PROOF. We shall use the following notation:

$$v = \{v(S)\}_{S\in\Theta}, \quad \widetilde{v} = \{v(S)_{S\in\Theta\backslash C}, \widetilde{v}(C)\}.$$

$x^{\widetilde{v}}(\theta) \in R^N$ is the unique (by Lemma 2) solution to (19) for the vector of coalitional values $\widetilde{v}$ and agreement probabilities $\theta = \{\theta_S\}_{S\in\Theta}$.

$\theta(x^{\widetilde{v}}) = \{\theta_S(x^{\widetilde{v}})\}_{S\in\Theta}$, where $\theta_S(x^{\widetilde{v}}) = 1|0$ if $\delta x_S^{\widetilde{v}} \geq v(S) \mid \delta x_S^{\widetilde{v}} < v(S)$.

In what follows, we show that ME $x^{\widetilde{v}}$ converges to the ME $x = x^v = x^v(\theta(x^v))$ when $\widetilde{v}$ approaches v.

If $v(S) \neq \delta x_S^v$ for all $S \in \Theta$, then, for sufficiently small $|\widetilde{v}(S) - v(S)|$,

$$\begin{aligned}
\delta x_S^{\widetilde{v}}(\theta(x^v)) &\approx \delta x_S^v > v(S) \Rightarrow \delta x_S^{\widetilde{v}}(\theta(x^v)) > v(S), \\
\delta x_S^{\widetilde{v}}(\theta(x^v)) &\approx \delta x_S^v < v(S) \Rightarrow \delta x_S^{\widetilde{v}}(\theta(x^v)) < v(S).
\end{aligned}$$

As x^v is a ME, then $x^{\widetilde{v}}(\theta(x^v))$ is also a ME and

$$x^{\widetilde{v}}(\theta(x^v)) \xrightarrow[\widetilde{v}(S)\to v(S)]{} x^v(\theta(x^v)) = x^v.$$

Consider now the case $\delta x_S^v = v(S)$ for all coalitions $S \in \Delta \subseteq \Theta$. By the same argument as in the ME existence proof, we can show the existence of a fixed point $\widetilde{\theta} \in \varphi_\Delta(\widetilde{\theta})$ in the game with coalitional values $\widetilde{v}$, where $\widetilde{\theta}_S = \theta_S(x^v)$ for all $S \in \Theta\backslash\Delta$ and φ_Δ is defined in (20). Then,

$$\widetilde{v}(S) \to v(S) \Rightarrow x^{\widetilde{v}}(\widetilde{\theta}) \to x^v(\widetilde{\theta}) = x^v(\theta(x^v)) = x^v,$$

where the last but one equality follows because $\widetilde{\theta}_S = \theta_S(x^v)$ for all $S \in \Theta\backslash\Delta$ and for all $S \in \Delta$,

$$\delta x_S^v = v(S) \Rightarrow \widetilde{\theta}_S(\delta x_S^v - v(S)) = \theta_S(x^v)(\delta x_S^v - v(S)).$$

Moreover, as $x^v = x^v(\widetilde{\theta})$ is a ME, then for all $S \in \Theta\backslash\Delta$ and a sufficiently small $|\widetilde{v}(S) - v(S)|$,

$$\begin{aligned}
\delta x_S^{\widetilde{v}}(\widetilde{\theta}) &\approx \delta x_S^v(\widetilde{\theta}) > v(S) \Rightarrow \delta x_S^{\widetilde{v}}(\widetilde{\theta}) > v(S) \quad \& \quad \theta_S = \widetilde{\theta}_S = 0, \\
\delta x_S^{\widetilde{v}}(\widetilde{\theta}) &\approx \delta x_S^v(\widetilde{\theta}) < v(S) \Rightarrow \delta x_S^{\widetilde{v}}(\widetilde{\theta}) < v(S) \quad \& \quad \theta_S = \widetilde{\theta}_S = 1.
\end{aligned}$$

Hence, $x^{\widetilde{v}}(\widetilde{\theta})$ is also a ME and

$$x^{\widetilde{v}}(\widetilde{\theta}) \xrightarrow[\widetilde{v}(S)\to v(S)]{} x^{v}(\widetilde{\theta}) = x^{v}.\blacksquare$$

PROOF. *Theorem 1*:

Existence: Let $\Theta = \{S \subseteq \mathcal{N} : \pi_S > 0\}$ and define the system of linear equations,

$$z_i = \delta_i z_i + \sum\nolimits_{S\in\mathcal{S}_i} \theta_S \pi_S \frac{\alpha_i}{\alpha(S)}(v(S) - \delta z_S), \quad \forall i \in \mathcal{N}. \tag{19}$$

By Lemma 2, there is a unique solution $z(\theta)$ to (19) for any fixed $\theta \in [0,1]^{\#\Theta}$. For a set of coalitions $\Delta \subseteq \Theta$, define the correspondence $\varphi_\Delta(\theta) : [0,1]^{\#\Theta} \rightrightarrows [0,1]^{\#\Theta}$ as follows,

$$\begin{array}{lll} \forall S \in \Delta, & \varphi_\Delta(\theta)_S & \begin{cases} = 1 \quad if \quad \delta z_S(\theta) < v(S), \\ \in [0,1] \quad if \quad \delta z_S(\theta) = v(S), \\ = 0 \quad if \quad \delta z_S(\theta) > v(S), \end{cases} \\ \forall S \in \Theta\backslash\Delta, & \varphi_\Delta(\theta)_S = \theta_S. & \end{array} \tag{20}$$

By definition, this correspondence has a closed graph and convex and non-empty images (i.e., $\forall\theta \in [0,1]^{\#\Theta}$, $\varphi(\theta)$ is a nonempty and convex subset of $[0,1]^{\#\Theta}$). Then, by Kakutani's fixed point theorem, there exists some $\widetilde{\theta}$ such that $\widetilde{\theta} \in \varphi_\Delta(\widetilde{\theta})$. By setting $\Delta = \Theta$, we obtain that $z(\widetilde{\theta})$ is a ME,

$$\begin{aligned} (1-\delta_i) z_i(\widetilde{\theta}) &= \sum\nolimits_{S\in\mathcal{S}_i} \widetilde{\theta}\pi_S \frac{\alpha_i}{\alpha(S)}(v(S) - \delta z_S(\widetilde{\theta})) \\ &= \sum\nolimits_{S\in\mathcal{S}_i} \mathbf{I}_{v(S)>\delta z_S(\widetilde{\theta})} \pi_S \frac{\alpha_i}{\alpha(S)}(v(S) - \delta z_S(\widetilde{\theta})), \quad \forall i \in \mathcal{N}. \end{aligned}$$

Uniqueness: We proceed by induction on the number of coalitions in the set $\Theta = \{S \subseteq \mathcal{N} : \pi_S > 0\}$.

1) Assume that only one coalition meets with positive probability, $\Theta = \{S\}$. Then, direct computation shows that there is a unique ME such that $x_i = 0$ for $i \notin S$ and

$$x_i = \frac{\alpha_i \pi_S v(S)}{(1-\delta_i)\sum_{k\in S} \frac{\alpha_k(1-\delta_k+\pi_S\delta_k)}{1-\delta_k}},$$

for each member i of the agreeing coalition S. It is easily verified that $\delta x_S < v(S)$ confirming that the agreement in S is rational.

2) Assume a unique ME for $\#\Theta = n$.

3) In order to prove a unique ME for $\#\Theta = n+1$ assume, for the sake of contradiction, two MEa, x and z, $x \neq z$, with the respective sets of active coalitions S^x and S^z,

$$\begin{aligned} S^x &= \{C \subseteq \mathcal{N} : \theta_C = 1, \quad \theta_C(v(C) - \delta x_C) \geq 0\}, \\ S^z &= \{C \subseteq \mathcal{N} : \theta_C = 1, \quad \theta_C(v(C) - \delta z_C) \geq 0\}. \end{aligned}$$

Note that $S^x \neq S^z$ as, otherwise, Lemma 2 implies $x = z$. Take a coalition $C \in S^x \backslash S^z$ (or $C \in S^z \backslash S^x$). By our existence results, for each value $\widetilde{v}(C) \in [0, v(C)]$, there is a ME $x^{\widetilde{v}}$. Set $x^{\widetilde{v}} = x$ if $\widetilde{v}(C) = v(C)$ (or $x^{\widetilde{v}} = z$ if $C \in S^z \backslash S^x$). Due to the continuity of $x^{\widetilde{v}}$ in $\widetilde{v}(C)$ (Lemma 3) and the ME condition, $0 \leq \delta x_C \leq v(C)$, there is a value $\widetilde{v}(C)$ such that $\delta x^{\widetilde{v}}_C = \widetilde{v}(C)$. Then, if we remove C with this value from Θ, i.e., if we set $\pi_C = 0$, $x^{\widetilde{v}}$ will remain a ME. On the other hand, the ME z does not change with $\widetilde{v}$ because $C \notin S^z$,

$$\delta z^{\widetilde{v}}_C = \delta z_C \geq v(C) \geq \widetilde{v}(C).$$

Then, in the game with $\#\Theta = n$ coalitions, we have two MEa, $x^{\widetilde{v}}$ and $z^{\widetilde{v}} = z$ which contradicts our inductive hypothesis in 2).

PROOF. *Theorem 2*:

(i) First, we show that $v(S) \leq x^1(S)$ for each coalition $S \subseteq \mathcal{N}$ in the LME x^1. If there existed a coalition S such that $v(S) > x^1(S)$, then $\pi_S > 0$ due to our assumption in Section 4 and, by re-arranging (1) for a player $i \in S$ and taking the limit, we would obtain a contradiction,

$$\underbrace{\lim_{d \to 1} (1-d) x_i^d}_{=0} = \underbrace{\lim_{d \to 1} \sum\nolimits_{S \in \mathcal{S}_i} \pi_S \frac{\alpha_i}{\alpha(S)} \max\{v(S) - d x^d(S), 0\}}_{>0}, \tag{21}$$

where $d = \delta_1 = ... = \delta_N$. Furthermore, if S is active in the LME x^1, $v(S) > dx^d(S)$ for all $d < 1$ sufficiently close to one (as follows from our discussion below the LME definition 1) and, hence, (21) implies $v(S) = x^1(S)$ for each active coalition S in the LME x^1.

Now, in order to prove LME payoff equality for players of the same type, we assume, for the sake of contradiction, that $x_i^1 > x_j^1$ for two distinct players of the same type, $T(\{i\}) = T(\{j\})$, with $i \in S$ and $j \notin S$ for a coalition S that is active in the LME x^1. If $S' := \{j\} \cup S \backslash \{i\}$, then $T(S) = T(S')$ and,

$$x^1(S) = v(S) = v(S') > x^1(S') = x^1(S) - x_i^1 + x_j^1,$$

which contradicts that $v(S') \leq x^1(S')$ for any coalition $S' \subseteq \mathcal{N}$ in the LME x^1. Hence, we conclude that $x_i^1 = x_j^1$ if $T(\{i\}) = T(\{j\})$.

(ii) We showed that $x^1(S) \geq v(S)$ (with equality of active coalitions) and that all players of the same type receive equal limit payoffs in (i). From these results, it follows directly that $\sum_t n_t x_t^1 = v(\mathbf{n})$ for each active coalition of type $\mathbf{n} = (n_1, \ldots, n_T)$.

(iii) Consider a player of type t who cooperates in a LME x^1 in a coalition S with $T(S) = \mathbf{n}$. Then, by (i), we can write

$$x^1(S) = v(S) = v(\mathbf{n}) = \sum\nolimits_{t=1}^{T} n_t x_t^1, \quad x^1(S') \geq v(S'), \quad \forall S' \subseteq \mathcal{N}.$$

The latter inequality must hold, in particular, for

$$S' = S^- : T(S^-) = \mathbf{n} - \mathbf{e}_t \quad \& \quad S' = S^+ : T(S^+) = \mathbf{n} + \mathbf{e}_t.$$

where $v(\mathbf{n} - \mathbf{e}_t)$ and $v(\mathbf{n} + \mathbf{e}_t)$ are well-defined values. Then,

$$x^1(S^-) = \sum\nolimits_{s \neq t} n_s x_s^1 + (n_t - 1)x_t = v(\mathbf{n}) - x_t \geq v(\mathbf{n} - \mathbf{e}_t) = v(S^-),$$
$$x^1(S^+) = \sum\nolimits_{s \neq t} n_s x_s^1 + (n_t + 1)x_t = v(\mathbf{n}) + x_t \geq v(\mathbf{n} + \mathbf{e}_t) = v(S^+),$$

which yields the claim.

(iv) First, by (i) we observe that in the LME x^1, $x^1(S) \geq v(S)$ for any $S \subseteq \mathcal{N}$. From this and from the definition of feasibility of the payoff profile $\{S^k, y_t^k\}_{t=1,..,T}^{k\in\mathcal{K}}$ follows that $x^1(S^k) \geq v(S^k) \geq y^k(S^k), \quad \forall k \in \mathcal{K}$. ■

Lemma 4. *In an PLME x^1,*

$$\forall S, S' \subseteq \mathcal{N} : T(S) \neq T(S'), S \cap S' \neq \varnothing, \quad x^1(S) = v(S) \Rightarrow x^1(S') > v(S').$$

PROOF. If $x^1(S) = v(S)$ and $x^1(S') \leq v(S')$ then, by the definition (6), $T(S) = T(S')$, which contradicts $T(S) \neq T(S')$. ■

Lemma 5. *If the ME x^δ implies that all players of types t and s cooperate only in coalitions of type $n = (n_1, ..., n_T)$, then,*

$$n_s \alpha_s (1 - \delta_t) x_t^\delta(\mathcal{N}_t) = n_t \alpha_t (1 - \delta_s) x_s^\delta(\mathcal{N}_s).$$

PROOF. By summing up (1) over all t-type players for some $t \in \{1, \ldots, T\}$, one obtains the total payoff of the players of this type,

$$\begin{aligned} x_t^\delta(\mathcal{N}_t) &= \delta_t x_t^\delta(\mathcal{N}_t) + n_t \alpha_t \sum\nolimits_{S:T(S)=\mathbf{n}} \pi_S \frac{v(S) - \delta x_S^\delta}{\alpha(S)} \\ &= \frac{n_t \alpha_t}{1 - \delta_t} \sum\nolimits_{S:T(S)=\mathbf{n}} \pi_S \frac{v(S) - \delta x_S^\delta}{\alpha(S)} =: \frac{n_t \alpha_t}{1 - \delta_t} \Delta^\delta(\mathbf{n}). \end{aligned}$$

By the same argument, the total payoff to the players of type $s \in \{1, \ldots, T\}$ is $x_s^\delta(\mathcal{N}_s) = n_s \alpha_s \Delta^\delta(\mathbf{n})/(1 - \delta_s)$ and the claim follows. ■

PROOF. *Theorem 3*:

(i) To show that in an PLME, all players of the same type cooperate in MCs of homogeneous types, we will proceed by establishing a contradiction. Consider an PLME x^1 and assume, for the sake of contradiction, that in x^1 players i and j of type t cooperate in coalitions S and S', respectively, with $T(S) = (n_1, ..., n_T) \neq (n'_1, ..., n'_T) = T(S')$. Let the coalition S'' be the same as S' except for player i who replaces player j, i.e. $S'' = S' \cup \{i\} \setminus \{j\}$. Hence, $T(S) \neq T(S')$ $= T(S'')$ and $\{i\} \in S \cap S''$. As S and S' are active, Theorem 2, items (i) and (ii), and the fact that $T(S') = T(S'')$ imply,

$$v(S) = x^1(S), \quad x^1(S') = v(S') = v(S'') = x^1(S'').$$

However, this contradicts, by Lemma 4, the partitioning property of x^1 as $S \cap S'' \neq \varnothing$ and $T(S) \neq T(S'')$.

(ii) Let $d = \delta_1 = ... = \delta_N$ and consider an PLME x^1. We have established that each player of type t cooperates in MCs of homogeneous types. By Lemma 5, the total payoff for all players of type t and all players of type s, that cooperate in MCs of the same type as a type $t-$player, i.e., $\mathbf{n} = (n_1, ..., n_T)$, satisfy

$$n_s \alpha_s x^d(\mathcal{N}_t) = n_t \alpha_t x^d(\mathcal{N}_s).$$

This equality must hold also for the PLME x^1. Thus using Theorem 2(i), we can re-write the last equality as

$$n_s \alpha_s N_t x^1_t = n_t \alpha_t N_s x^1_s. \tag{22}$$

In particular, $x^1_t = 0$ implies $x^1_s = 0$ for any two distinct types that cooperate in a coalition $S : T(S) = \mathbf{n}$. This is only possible in a PLME if $v(S) = v(\mathbf{n}) = 0$. Suppose $x^1_t > 0$ for some type t. Then, by Theorem 2 items (i) and (ii), it follows that

$$v(S) = v(\mathbf{n}) = x^1(S) = \sum\nolimits_{s=1}^{T} n_s x^1_s.$$

Using (22) to substitute for x^1_s,

$$v(\mathbf{n}) = \sum\nolimits_{s=1}^{T} \frac{n_s^2 \alpha_s N_t x^1_t}{n_t \alpha_t N_s} = x^1_t \frac{N_t}{n_t \alpha_t} \sum\nolimits_{s=1}^{T} \frac{n_s^2 \alpha_s}{N_s}.$$

By re-arranging the above expression, we obtain the payoff of each type-t player in the PLME x^1,

$$x^1_t = v(\mathbf{n}) \frac{n_t \alpha_t / N_t}{\sum_{s=1}^{T} (n_s^2 \alpha_s / N_s)}. \qquad \blacksquare$$

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