MIRROR UTILITY FUNCTIONS AND REFLEXION PROPERTIES OF VARIOUS CLASSES OF GOODS

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Abstract Any 2-good direct utility function satisfying standard axioms may be transformed into an indirect utility function, also satisfying standard axioms, by a straightforward change of sign. The reverse is also true. We shall refer to one such function as the 'mirror' of the other. It is sometimes the case that the demand function for one of the goods, arising from one utility function, exhibits a particular feature if and only if the mirror utility function exhibits the same feature for the *other* good. When this occurs, we say that the demand feature in question has the 'reflexion property'. It is shown that Giffen behaviour and the necessity/luxury dichotomy are two features of demand that do have this reflexion property. However, it is also shown that the normality/inferiority dichotomy is one feature that does not.

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1 Introduction

A direct utility function (DUF) is a function whose arguments are the quantities consumed of different goods, and two of its basic properties are (increasing) monotonicity and quasi-concavity. An indirect utility function (IUF) is a function whose arguments are the normalised prices of the goods, and the corresponding properties are (decreasing) monotonicity and quasi-convexity. For both types of function, the indifference curves (i.e. the contours) are convex to the origin. These well-known observations suggest a simple method for obtaining one utility function from another: reversing the sign. Reversing the sign of a DUF that satisfies the basic axioms of consumer theory gives rise to an IUF that also satisfies basic axioms, and vice versa. We shall refer to such a pair of functions as a 'mirror pair'.

This coupling of a DUF with its mirror IUF leads to the question of what is the link, if any, between the preferences represented by the two functions. In particular, do the demand functions implied by the two utility functions have features in common? For some features, the answer turns out to be affirmative. For example, in the two-good context, we shall prove that good 1 is a 'Giffen' good (i.e. a good having a positive own-price effect) under a given DUF if and only if good 2 is a Giffen good under the mirror IUF. Because of this, we shall say that the Giffen feature of demand has the 'reflexion' property. We shall further prove that good 1 is a luxury (i.e. a good whose share of income rises as income rises) under the DUF if and only if good 2 is a luxury under the mirror IUF. However, not all features of the demand have the reflexion property: we shall show that if a good is inferior under one utility function (i.e. its demand falls as income rises), it does not follow that the other good will be inferior under the mirror function.

One reason for the usefulness of this sort of result is that it can provide a means of obtaining a utility function with desired demand features. One example has been the subject of recent intense interest: ever since Wold & Jureen (1953), there has been a search for a DUF represented by a single smooth function over the whole positive (x, y) quadrant that exhibits Giffen behaviour and satisfies all of the axioms of consumer theory (see e.g. Spiegel 1994, Weber 1997, 2001, Doi, Iwasa & Shimomura 2009). A means of constructing such functions was devised by Moffatt (2002), and explicit examples have been obtained by Sørenson (2007), Heijman & von Mouche (2011) and Moffatt (2011). In the last of these cases (to be considered further in §3.3 below), the DUF was obtained by first finding an IUF with the desired properties, and then invoking the reflexion property of Giffen behaviour, anticipated on the basis of geometrical arguments.

The reflexion property is also potentially useful in the opposite direction. Suppose that I am working with a particular DUF, by which certain desirable demand characteristics are implied,and that I wish to estimate the demand parameters empirically. Except in the simplest settings, the demand functions cannot be deduced from the DUF in closed form, preventing direct estimation of the parameters. The solution is to invoke the reflexion property (when this applies to the demand features of interest) to obtain an IUF with the same demand features. Then, Roy's identity may be used to obtain Marshallian (i.e. estimable) demand functions with these desired features.

The geometry of contour maps of both direct and indirect utility functions play a central role in this paper; the methods adopted are in fact motivated by geometrical considerations, which are then readily translated into analytical results. The plan of the paper is as follows. In §2, we recapitulate some standard results concerning convexity, which play an essential role in the subsequent analysis. In §3, we obtain criteria for Giffenity both from the standpoint of an IUF, and from that of a DUF, and we prove the reflexion property, as described above, of a mirror pair of such functions (see Theorem 1 in §3.2). The result is contrasted with the result of Kohli (1985) concerning the relationship between the slopes of direct and inverse demand functions in the two-good situation. In §4, we obtain criteria for inferiority of either good, again from both standpoints, and we provide several examples to demonstrate that there is no obvious reflexion property associated with this demand feature. The relationship between Giffenity and inferiority is nevertheless illuminated by the formulae obtained relating 'Giffen functions' and 'inferiority functions' from both IUF and DUF standpoints. Finally, in §5, a criterion for one of the goods to be a luxury good is obtained, and again, a reflexion property is established (see Theorem 2 in §5.3). We conclude in §6 with a summary of the essential results obtained, and a discussion of possible future lines of enquiry.

2 Preliminary considerations

Let $U(x,y)$ be the (direct) utility function for a two-good situation with consumption variables (x,y). We assume that $U(x,y)$ satisfies the basic axioms of consumer theory, namely the monotonicity conditions

$$U_x \equiv \frac{\partial U}{\partial x} > 0, \quad U_y \equiv \frac{\partial U}{\partial y} > 0\,, \tag{1}$$

and the condition that the contours $U(x,y) = \text{const.}$ be convex to the origin (i.e. that U be quasi-concave), a condition that may be expressed in the 'bordered Hessian' form (see for example Takayama 1985, chap.1(E,F))[1]

$$C(x,y) \equiv \frac{1}{U_y^3}\begin{vmatrix} 0 & U_x & U_y \\ U_x & U_{xx} & U_{xy} \\ U_y & U_{xy} & U_{yy}\end{vmatrix} \equiv \frac{2U_xU_yU_{xy} - U_x^2U_{yy} - U_y^2U_{xx}}{U_y^3} > 0\,. \tag{2}$$

Suffixes will throughout denote partial differentiation. Note that, if we replace $U(x,y)$ by any function of the form $\hat{U}(x,y) = F(U(x,y))$, with $F'(U) > 0$ (e.g. $\hat{U} = -U^{-1}$), then the monotonicity conditions (1) remain satisfied; moreover, the 'convexity function' $C(x,y)$ is invariant under such functional changes, as may be easily verified; i.e. $C(x,y)$ is a property of the geometry of the contour map, but not of the values of U on the various contours.

Let $A(x,y), B(x,y)$ be defined by[2]

$$A(x,y) \equiv U_yU_{xy} - U_xU_{yy}, \quad B(x,y) \equiv U_xU_{xy} - U_yU_{xx}\,. \tag{3}$$

The inequality (2) may then be rearranged in the alternative form

$$C(x,y) \equiv \frac{AU_x + BU_y}{U_y^3} > 0\,. \tag{4}$$

Equivalently, if the equation of any contour is $y = Y(x) > 0$, then $Y'(x) < 0$ and $Y''(x) > 0$ for all $x > 0$.

By way of example, consider the special class of utility functions of the form

$$U(x,y) = yf(x)\,, \tag{5}$$

with $f(x) > 0$ and $f'(x) > 0$ for $0 < x < \infty$. The contours $U = c$ are then the family of curves

$$y = c/f(x)\,, \tag{6}$$

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[1] The curvature $\kappa(x,y)$ (here positive) of the contour $U(x,y)$ =const. is related to $C(x,y)$ by the formula $\kappa(x,y) = U_y^3(U_x^2 + U_y^2)^{-3/2}C(x,y)$; the radius of curvature is $R = \kappa^{-1}$.

[2] We note that $(-A)$ and B are minors of the determinant in (2). The notation adopted here is deliberate for reasons of symmetry; but note that $B(y,x) \equiv A(x,y)$ if and only if $U(y,x) \equiv U(x,y)$.

for constants $c > 0$. We then have

$$dy/dx = -cf'/f^2, \tag{7}$$

and the convexity condition requires that

$$\frac{d^2y}{dx^2} = \frac{c}{f^3}(2f'^2 - ff'') > 0\,. \tag{8}$$

For U given by (5), we have also

$$U_x = yf'(x), \quad U_y = f(x), \quad U_{xx} = yf''(x), \quad U_{xy} = f'(x), \quad U_{yy} = 0\,, \tag{9}$$

and so, from (3),

$$A(x,y) = ff', \quad B(x,y) = y(f'^2 - ff'')\,, \tag{10}$$

and (4) reduces to

$$AU_x + BU_y = yf(f'^2 - ff'') + yff'^2 = yf(2f'^2 - ff'') > 0\,, \tag{11}$$

in agreement, as expected, with (8).

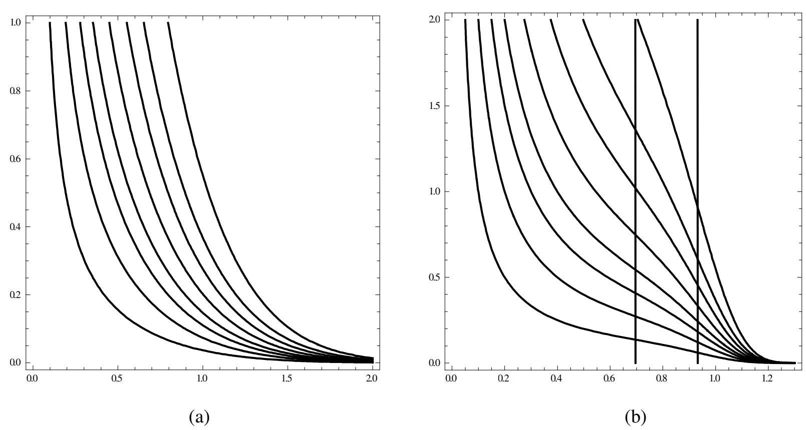

Fig. 1 Contours of (a) the quasi-concave utility function $U(x,y) = y\,x\exp(x^2)$, and (b) the non-quasi-concave function $U(x,y) = y\,x\exp(x^8)$; the region of non-concavity lies between the two vertical lines ($x = 0.696$ and $x = 0.932$), which pass through inflexion points on all the contours.

We shall be particularly concerned with the case for which

$$f(x) = x\exp(x^n)\,, \tag{12}$$

for integer $n \geq 0$, which obviously satisfies the condition $f'(x) > 0$, and which might be considered to provide a useful candidate for detailed study. Care is needed however: for, from (12), we have

$$f'(x) = (1 + nx^n)\exp(x^n), \quad f''(x) = [(n^2 + n)x^{n-1} + n^2x^{2n}]\exp(x^n)\,, \tag{13}$$

and so

$$2f'^2 - ff'' = [2 + (3n - n^2)x^n + n^2x^{2n}]\exp(2x^n)\,; \tag{14}$$

this is positive for all $x > 0$ only[3] for positive integer n in the range

$$0 \leq n \leq 5\,. \tag{15}$$

The corresponding $U(x, y)$ is therefore quasi-concave only for n in this range. Figure 1 shows the contours for two contrasting cases $n = 2$ and $n = 8$. The reason for non-quasi-concavity in the latter case is evidently that for x greater than about 0.7, the gentle descent of the contours $\sim x^{-1}$ gives way to the much steeper descent associated with the factor $\exp(-x^8)$, and this transition is sufficiently sharp for the contours to be concave to the origin over a range of x (actually $0.696 < x < 0.932$ in this case).

3 Giffen goods

3.1 Giffenity in terms of the indirect utility function

Let us now consider certain properties of a general indirect utility function (IUF) $V(p, q)$, a function of the prices (p, q) conjugate to (x, y), with normalised budget constraint

$$px + qy \leq 1\,. \tag{16}$$

We first address the question: how can we detect, from the functional form of $V(p, q)$, whether either good exhibits Giffen behaviour (or 'Giffenity'), i.e. an increase in the quantity demanded of a good in response to an increase in its own price (the price of the other good being held fixed)? It is obvious from the constraint (16) that at most one of the two goods can exhibit Giffenity at any particular values (P, Q) of (p, q).

First note that $V(p, q)$ must satisfy the negative gradient conditions

$$V_p \equiv \frac{\partial V}{\partial p} < 0\,, \quad V_q \equiv \frac{\partial V}{\partial q} < 0\,. \tag{17}$$

[3] The results apply equally if n is any positive real number, the upper limit for quasi-concavity then being $n = 3 + 2\sqrt{2} \approx 5.828$.

Moreover, as explained by Suen (1992), the contours $V =$ const. are, like those of any 'standard' *direct* utility function, convex to the origin. By analogy with (2), this requirement may be expressed in the form

$$C(p,q) \equiv \frac{2V_pV_qV_{pq} - V_p^2V_{qq} - V_q^2V_{pp}}{V_q^3} \; > \; 0\,, \tag{18}$$

an inequality that must be satisfied at all points of the positive (p,q) quadrant. Note again that the conditions (17) and (18) are invariant under any change of IUF of the form $\hat{V}(p,q) = F(V(p,q))$ with $F'(V) > 0$, and are therefore properties of the geometry of the contour map, but not of the values of V on the various contours.[4]

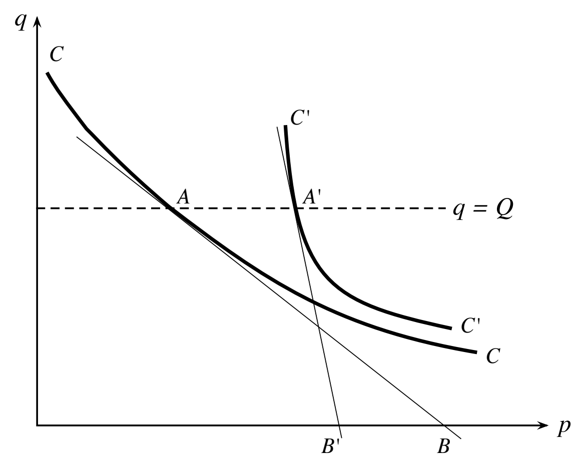

Fig. 2 The curves C and C' are contours $V = c$ and $V = c'$ of an indirect utility function $V(p,q)$. The line $q = Q$ intersects C at A, C' at A'. AB is tangent to C and $A'B'$ is tangent to C'. As A moves to the right across intermediate contours towards A', B moves to the left towards B', a symptom of Giffenity in good 1.

With reference to figure 2, let A be a point with coordinates (P,Q) on the contour $C\colon V(p,q) = c$. On this contour,

$$dV \equiv V_p dp + V_q dq = 0\,, \tag{19}$$

and so the gradient of the tangent to the contour at A is $m(P,Q)$, where

$$m(p,q) = \left[\frac{dq}{dp}\right]_{V=const.} = -\,\frac{V_p}{V_q}\,. \tag{20}$$

[4] Since both numerator and denominator in the expression for C are cubic in V, the inequality (18) persists even under change of sign relating U and V.

Clearly $-\infty < m < 0$. The convexity condition (18) is of course just an expression of the fact that m increases as the point (p,q) moves along the contour in the direction of increasing p, i.e.

$$\left[\frac{dm}{dp}\right]_{V=const.} \equiv \frac{\partial m}{\partial p} + \frac{\partial m}{\partial q}\left[\frac{dq}{dp}\right]_{V=const.} = \frac{\partial m}{\partial p} - \frac{V_p}{V_q}\frac{\partial m}{\partial q} \;>\; 0\,, \qquad (21)$$

an inequality from which the condition (18) may be most simply and directly obtained; indeed, as may be verified,

$$\frac{\partial m}{\partial p} - \frac{V_p}{V_q}\frac{\partial m}{\partial q} \equiv C(p,q)\,. \qquad (22)$$

The budget line (with budget normalised to unity)

$$px + qy = 1 \qquad (23)$$

touches the contour $V(p,q) =$ const. at the point A where their gradients coincide, i.e. where $y/x = V_Q/V_P$, or equivalently where

$$xV_Q - yV_P = 0\,. \qquad (24)$$

Solving (23) and (24) for x and y gives Roy's identity

$$x(P,Q) = \frac{V_P}{PV_P + QV_Q}, \quad y(P,Q) = \frac{V_Q}{PV_P + QV_Q}\,. \qquad (25)$$

The equation of the tangent at A is

$$q - Q = m(P,Q)(p-P)\,, \qquad (26)$$

and this intersects the p-axis at the point B with coordinates $(g,0)$, where

$$g(P,Q) = P - \frac{Q}{m(P,Q)} = P + Q\frac{V_Q}{V_P} = \frac{PV_P + QV_Q}{V_P}\,. \qquad (27)$$

Thus, from (25), we have immediately

$$g(P,Q) = \frac{1}{x(P,Q)}\,. \qquad (28)$$

Now the condition for Giffenity of good 1 at the point A is that $x(P,Q)$ should be an increasing function of P (and so $g(P,Q)$ a decreasing function of P) for constant Q, i.e. that, as A moves to the right across adjacent contours, B should move to the left on the p-axis, as illustrated in figure 2 . Thus the condition for Giffenity of good 1 at the point A is that the function $G(P,Q)$, defined by

$$G(P,Q) \equiv \frac{\partial g}{\partial P} = \frac{V_P^2 + Q(V_P V_{PQ} - V_Q V_{PP})}{V_P^2}\,, \tag{29}$$

should be negative:

$$G(P,Q) \equiv \frac{V_P^2 + Q(V_P V_{PQ} - V_Q V_{PP})}{V_P^2} = 1 + \frac{Q\,B(P,Q)}{V_P^2} \;< 0\,. \tag{30}$$

Similarly, the condition for Giffenity of good 2 is that

$$H(P,Q) \equiv \frac{V_Q^2 + P(V_Q V_{PQ} - V_P V_{QQ})}{V_Q^2} = 1 + \frac{P\,A(P,Q)}{V_Q^2} \;<\; 0\,. \tag{31}$$

Note that, despite appearances, $H(P,Q) \neq G(Q,P)$, because in general $V(P,Q) \neq V(Q,P)$.

Let $\mathcal{G}_1(p,q)$ be the region of the positive (p,q) quadrant (if any) in which $G(p,q) < 0$, i.e. in which good 1 is a Giffen good. Similarly, let $\mathcal{G}_2(p,q)$ be the region (if any) in which $H(p,q) < 0$, i.e. in which good 2 is a Giffen good. It is clear that at most one of the two goods can exhibit Giffenity at the same point (P,Q); hence these regions are non-overlapping. Moreover the product $G(P,Q)H(P,Q)$ is negative if and only if one of the goods is a Giffen good at (P,Q), i.e. iff $(P,Q) \in \mathcal{G}(p,q) = \mathcal{G}_1(p,q) \bigcup \mathcal{G}_2(p,q)$. The 'Giffen-region' $\mathcal{G}(p,q)$ is then the region (if any) of the positive (p,q) quadrant in which the inequality

$$G(p,q)H(p,q) < 0 \tag{32}$$

is satisfied. At any point in this region, one of the two goods is a Giffen good.

3.2 Giffenity in terms of the direct utility function

We may now seek a similar criterion for Giffenity in terms of a direct utility function $U(x,y)$. Consistent with (30) above, we first *define* $G(X,Y)$ by the equation

$$G(X,Y) = \frac{U_X^2 + Y(U_X U_{XY} - U_Y U_{XX})}{U_X^2} = 1 + \frac{Y B(X,Y)}{U_X^2}\,. \tag{33}$$

Now, much as before, consider the tangent at the point A (coordinates (X,Y)) of a contour $U(x,y) = c$ (see figure 3 below); this meets the x-axis at the point B (coordinates $(g,0)$), where now (cf eqn. (27))

$$g(X,Y) = X + Y\frac{U_Y}{U_X}\,. \tag{34}$$

If this tangent is the budget line

$$px + qy = 1 \; (= pX + qY)\,, \tag{35}$$

with gradient $m_b = -p/q$, then this gradient is equal to $-U_X/U_Y$ at the point (X, Y), i.e.

$$p\,U_Y - q\,U_X = 0\,. \tag{36}$$

Consider now what happens if we increase the price q of good 2, keeping the price p of good 1 (and the budget) fixed; this merely changes the gradient m_b while keeping $g = 1/p$ fixed. Thus

$$dg \equiv \frac{\partial g}{\partial X} dX + \frac{\partial g}{\partial Y} dY = 0\,. \tag{37}$$

Also, from(23), the point of contact (X, Y) of the budget line varies according to

$$pdX + qdY + Ydq = 0\,. \tag{38}$$

Hence, eliminating dX, (37) and (38) give

$$\frac{dY}{dq} = \frac{Y\partial g/\partial X}{p\partial g/\partial Y - q\partial g/\partial X}\,. \tag{39}$$

Now, from (34), we have

$$\frac{\partial g}{\partial X} = 1 + Y\frac{U_X U_{XY} - U_Y U_{XX}}{U_X^2} = G(X,Y)\,, \tag{40}$$

and, using (2),

$$\frac{\partial g}{\partial Y} = \frac{U_Y}{U_X} + Y\frac{U_X U_{YY} - U_Y U_{XY}}{U_X^2} = \frac{U_Y(U_X^2 G(X,Y) - YU_Y^2 C(X,Y))}{U_X^3}\,. \tag{41}$$

Thus (39) gives

$$\frac{dY}{dq} = \frac{YU_X^3 G(X,Y)}{-pYU_Y^3 C + U_X^2(pU_Y - qU_X)G(X,Y)} = \frac{-U_X^3 G(X,Y)}{pU_Y^3 C(X,Y)}\,, \tag{42}$$

using (2). It follows immediately (using (1), (2) and $p > 0$) that

$$G(X,Y) < 0 \quad \Longleftrightarrow \quad dY/dq > 0\,. \tag{43}$$

Thus $G(X, Y) < 0$ is now a necessary and sufficient condition for Giffenity of good 2. Similarly, $H(X, Y) < 0$ is a necessary and sufficient condition for Giffenity of good 1.

These results motivate introduction of the concept of a ‘mirror’ pair of utility functions (direct and indirect) $\{U(x, y), V(p, q)\}$ related by the substitutions $x \leftrightarrow p, y \leftrightarrow q$ and a simple change of sign:

$$V(p,q) = -U(p,q), \quad \text{or equivalently} \quad U(x,y) = -V(x,y)\,, \qquad (44)$$

so that the monotonicity conditions (1) and (17) are compatible, and the contour maps of U and V identical.[5] For such a mirror pair, we have proved the following theorem:

Theorem 1. If good 1 associated with an indirect utility function $V(p,q)$ is a Giffen good in a region $\mathcal{G}_1(p,q)$ of the positive (p,q) quadrant then good 2 associated with the mirror direct utility function $U(x,y)\,(= -V(x,y))$ is a Giffen good in the corresponding region $\mathcal{G}_1(x,y)$ of the positive (x,y) quadrant; and conversely.

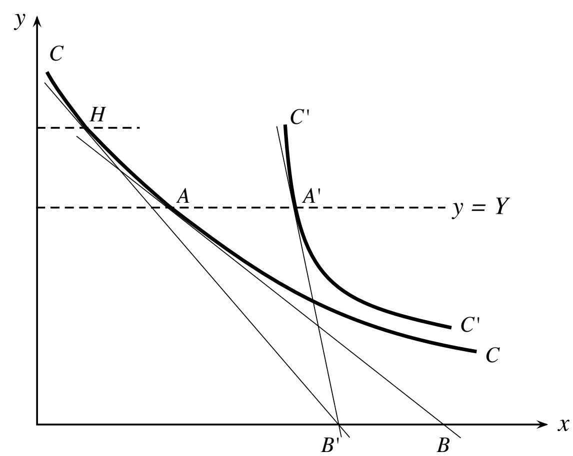

Fig. 3 The same as figure 2, but now in the (x,y)–plane. $B'H$ is now the tangent from B' to C, and it is evident that, as a result of the convexity of C, H is above A on C, i.e. $y_H > y_A\,(= y_{A'})$.

The result has a simple geometrical interpretation – see figure 3, which shows two adjacent contours C and C' with tangents AB, $A'B'$, and with $Y_A = Y_{A'}$; the Giffenity condition $G < 0$ means that B' is to the left of B, as shown. As B moves to the left along the x-axis towards B', the point of contact of the tangent from B to C moves up the curve from A by virtue of the convexity condition, and reaches the point H when B reaches B'. When B reaches B' therefore, $X_H < X_A$ and $Y_H > Y_A = Y_{A'}$. This geometrical argument, which implicitly uses both conditions $C > 0$ and $G < 0$, indeed

[5] Needless to say, the mirror function of an DUF $U(x,y)$ should not be confused with its conventional dual IUF $V_{dual}(p,q) = U(x(p,q), y(p,q))$, where $x(p,q)$ and $y(p,q)$ are Marshallian demand functions.

confirms Giffenity in good 2. The argument is reversible, confirming that the conditions $Y_H > Y_{A'}$ and $C > 0$ together imply that $G < 0$.

If neither good is a Giffen good, then the demand functions for both goods slope downwards (i.e. satisfy the 'law of demand', a situation that has been discussed and extended to the multi-good context by Quah (2000, 2003). A corollary of our Theorem 1 is that if the law of demand is satisfied (in the two-good situation) by a DUF $U(x, y)$, then it is also satisfied by the mirror IUF $V(p, q) = -U(p, q)$. It seems likely that this result may be generalised to the multi-good situation; the papers of Quah provide a possible framework for such generalisation.

3.3 Relation with theorem of Kohli(1985)

We give here a compact proof of a result obtained by Kohli (1985), which should not be confused with Theorem 1 above. This result (equation (53) below) provides, in a two-good situation, a relation between the slope of the demand function of either good and the slope of the inverse demand function of the other good. [6]

For given $V(p, q)$, Roy's identity gives x and y explicitly as functions of p and q:

$$x = x(p, q) = \frac{V_p}{pV_p + qV_q}, \quad y = y(p, q) = \frac{V_p}{pV_p + qV_q}\,. \tag{45}$$

These equations may in principle be inverted to give p and q as functions of x and y:

$$p = p(x, y), \quad q = q(x, y)\,. \tag{46}$$

From (45), we have

$$dx = x_p dp + x_q dq, \quad dy = y_p dp + y_q dq\,. \tag{47}$$

Solving for dp, dq, we then have

$$dp = K^{-1}(y_q dx - x_q dy), \quad dq = -K^{-1}(y_p dx - x_p dy)\,, \tag{48}$$

where

$$K = x_p y_q - x_q y_p\,. \tag{49}$$

Now with A, B as defined by (3), x_p is given from (45) by

$$(pV_p + qV_q)^2 x_p = q(V_q V_{pp} - V_p V_{pq}) - V_p^2 = -B(p, q)q - V_p^2\,, \tag{50}$$

and similarly,

[6] We are grateful to the referee, whose comment led to the inclusion of this proof.

$$(pV_p + qV_q)^2(x_q, y_p, y_q) = (Aq - V_pV_q,\ Bp - V_pV_q,\ -Ap - V_q^2)\,. \tag{51}$$

Hence from (49), with convexity function $C > 0$ given by (4), K may be readily calculated in the form

$$K = py^{-3}C(p,q) > 0\,. \tag{52}$$

Now from (47), we have

$$p_x = K^{-1}y_q, \quad q_y = K^{-1}x_p\,. \tag{53}$$

Hence if y is a Giffen good (i.e. $y_q(p,q) > 0$), then $p_x(x,y) > 0$ also, i.e if the quantity of the good y is held constant, then p is an increasing function of x. This is the result of Kohli (1985) who discussed it in the context of the Irish potato famine of the 1840s. Kohli gave a geometrical interpretation that may be seen equally in our figure 3, for which as described in the text, $y_q > 0$, i.e. y is a Giffen good (in a finite region of the (p,q) quadrant). The point B in this figure has coordinates $(0, p^{-1})$, and it is evident that in this same region p^{-1} decreases, i.e. p increases, as x increases, as indeed implied by the first of equations (53).

3.4 Example

In a recent article (Moffatt 2011), we have constructed the following IUF (symmetric in p and q), and verified that it exhibits a region of Giffenity:

$$V(p,q) = \frac{S(p,q) - p - q + 2\lambda(1-\lambda)}{2(1-\lambda^2)}, \tag{54}$$

where

$$S(p,q) = \sqrt{[p + q - 2\lambda(1-\lambda)]^2 - 4(1-\lambda^2)[pq - (1-\lambda)^2]}\,. \tag{55}$$

Here, λ is a parameter which can be varied between zero and one, and the region of Giffenity appears when λ is greater than approximately 0.7. Figure 4 shows the contour map of the IUF (54) when $\lambda = 0.8$. The essential feature is that the indifference curves (i.e. the contours, all hyperbolic in form) become more strongly curved with distance along the diagonal from the origin as far as the point $(\lambda^{-1}, \lambda^{-1})$, here $(1.25, 1.25)$. (The parameter λ represents the rate at which the curvature increases on this portion of the diagonal. Beyond the point $(\lambda^{-1}, \lambda^{-1})$ on the diagonal, the curvature decreases, and there is no Giffenity in the region $(x > \lambda^{-1}) \bigcap (y > \lambda^{-1})$.)

Tangents to two adjacent contours at the same value of $q\,(= 1.5)$ are shown in figure 4. As in figure 2, these tangents intersect above the axis $q = 0$, an indication of Giffenity of good 1, as explained in §2. The region of Giffenity

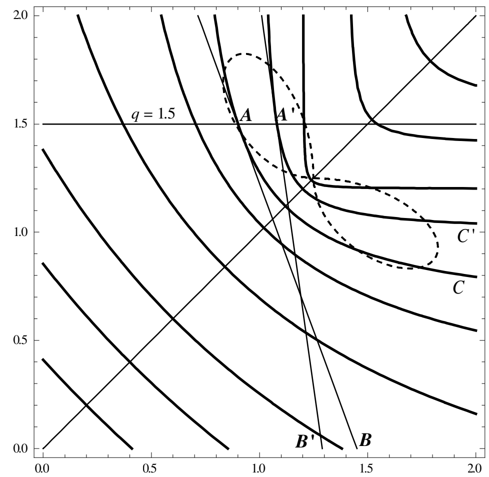

Fig. 4 Contours of the IUF $V(p,q)$, given by (54) with $\lambda = 0.8$; the Giffen regions $\mathcal{G}_1(p,q)$ and $\mathcal{G}_2(p,q)$ are inside the closed curves (dashed) above and below the diagonal respectively. These meet at the singular point $(1.25, 1.25)$. Tangents to adjacent contours C and C' at the same value of $q\,(=1.5)$ at the points A and A' (both in $\mathcal{G}_1(p,q)$) intersect above the p-axis (just as in figure 2) indicating Giffenity of good 1.

of good 1 is the region inside the dashed loop above the diagonal. By the symmetry of the IUF (54), the region of Giffenity of good 2 is obtained by reflexion in the diagonal $p = q$.

The mirror DUF is given by

$$U(x,y) = \frac{-S(x,y) + x + y - 2\lambda(1-\lambda)}{2(1-\lambda^2)}, \tag{56}$$

where now

$$S(x,y) = \sqrt{[x+y-2\lambda(1-\lambda)]^2 - 4(1-\lambda^2)[xy-(1-\lambda)^2]}\,. \tag{57}$$

The contour map of the DUF (56) again for $\lambda = 0.8$ is of course the same as in figure 4, but is redrawn again in figure 5 with a different purpose. As in figure 3 above, we here draw tangents to two adjacent contours from a point B' on the axis $q = 0$; these represent budget lines before and after a rise in the price q of good 2. It is evident that the budget line on the left has a higher point of tangency ($y_H = 1.61$) than that on the right ($y_{A'} = 1.5$), indicating Giffenity of good 2. The region of Giffenity of good 2 in figure 5 is identical with that of good 1 in figure 4, as indeed it must be by virtue of Theorem 1.

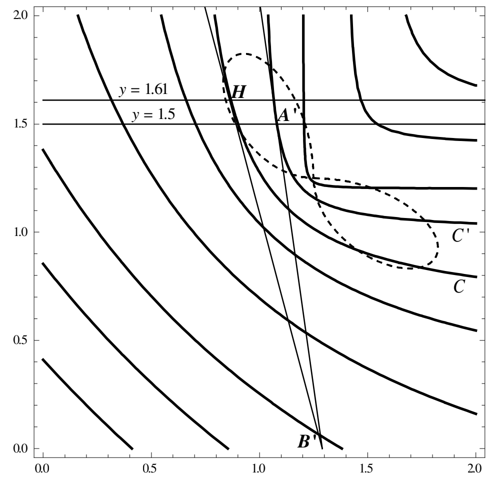

Fig. 5 Contour map of DUF (56) with $\lambda = 0.8$. Tangents $B'A', B'H$ are drawn from the point B' on the p-axis to two adjacent contours C and C'; as the budget line rotates anti-clockwise about B', the point of tangency rises from A' to H (cf. figure 3), indicating that good 2 is a Giffen good. The regions of Giffenity are identical with those of figure 4.

4 Normal and inferior goods

4.1 Reflexion properties in general

The foregoing 'reflexion' result for Giffenity of mirror pairs $\{U(x,y), V(p,q)\}$ leads naturally to the question as to whether there may be analogous results for other types of goods. In this section, we explore this question for the case of the normality/inferiority dichotomy. An 'inferior' good is one whose consumption decreases with increasing budget b, the unit prices p and q of the two goods remaining fixed; otherwise the good is 'normal'. Clearly if one good is inferior, the other must be normal. This is illustrated in figure 6, which shows contours of a DUF that exhibit inferiority of good 2 : the downward slope of the line AB indicates that the consumption y of good 2 decreases as b increases.

These tangents meet the y-axis in the points A' and B'. If we slide the lower tangent round its contour until it reaches the position EB', then in the case illustrated, $y_E < y_B$, so good 2 is not a Giffen good. Thus inferiority does not imply Giffenity, but, by a similar argument, Giffenity *does* imply inferiority. An analytical proof of this well-known result will be given below.

It is not clear *a priori* whether the reflexion property of Giffen goods will carry over to inferior goods. In fact we shall show that if a DUF $U(x,y)$ is

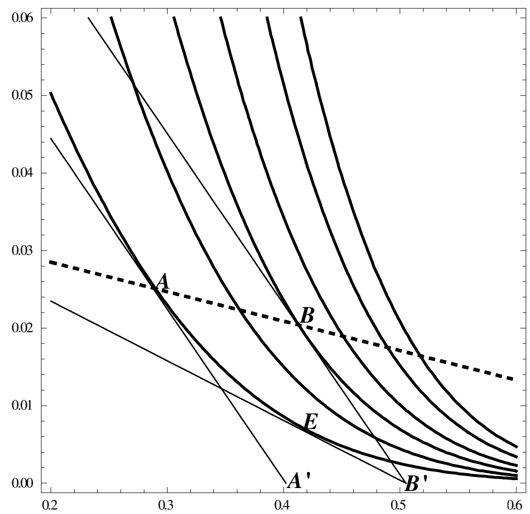

Fig. 6 Contour map for the DUF $U(x, y) = y \exp((x+1)^4)$. As the budget line AA' moves outwards, the y-coordinate of the point of tangency decreases ($y_B < y_A$), indicating that good 2 is inferior. If the budget line AA' slides round the contour to which it is tangent to the position EB', then $y_E < y_B$, so good 2 is not a Giffen good: inferiority does not imply Giffenity.

such that good 1, say, is inferior, then there is no guarantee that the mirror IUF $V(p, q)$ has a similar property; as we shall show, for some such DUFs, the mirror $V(p, q)$ is such that neither good is inferior, while for others, the mirror $V(p, q)$ is such that good 2 *is* inferior. We shall also provide a converse type of example of an IUF for which good 2 is inferior in a region of the (p, q) quadrant, but whose mirror DUF exhibits no inferiority in the corresponding region of the (x, y) quadrant.

4.2 Inferior goods from the IUF $\boldsymbol{V(p, q)}$

An increase of b at fixed (p, q) is equivalent to keeping b normalised to unity and therefore fixed, and decreasing p and q in equal proportions. This is equivalent to moving on the straight line $q = kp$ towards the origin, on which

$$dp/p = dq/q\,. \tag{58}$$

Just as in §3.1, $x(p,q)$ is given by Roy's identity in the form

$$g(p,q) = \frac{1}{x(p,q)} = p + q\frac{V_q}{V_p}\,, \tag{59}$$

so the change in g corresponding to price increments (dp, dq) is

$$dg = dp + \left(\frac{V_q}{V_p}\right) dq + q\frac{V_p dV_q - V_q dV_p}{V_p^2}\,. \tag{60}$$

With

$$dV_p = V_{pp}dp + V_{pq}dq, \quad \text{and} \quad dV_q = V_{pq}dp + V_{qq}dq\,, \tag{61}$$

this yields

$$[dg/dp]_{p/q=const.} = I(p,q)\,, \tag{62}$$

where

$$I(p,q) = 1 + \frac{qV_q}{pV_p} + \frac{q}{V_p^2}(V_pV_{qp} - V_qV_{pp}) + \frac{q^2}{pV_p^2}(V_pV_{qq} - V_qV_{pq})\,. \tag{63}$$

This may be manipulated to the form

$$I(p,q) = \left(1 + \frac{qV_q}{pV_p}\right) G(p,q) - \frac{q^2V_q^3}{pV_p^3}C(p,q)\,, \tag{64}$$

thus expressing $I(p,q)$ in terms of the Giffen function G and the (positive) convexity function C. Good 1 is inferior if x decreases (i.e. g increases) as p decreases (with q decreasing in proportion), i.e. if $[dg/dp]_{p/q=const.} < 0$. Hence the condition for good 1 to be inferior is

$$I(p,q) < 0\,. \tag{65}$$

Here, the expression (64) is illuminating: since $C(p,q) > 0$, if $G(p,q) < 0$ then certainly $I(p,q) < 0$ also, i.e. Giffenity implies inferiority. However, the converse is clearly not true. Equation (64) embodies the analytical counterpart of the geometrical argument presented in §4.1 above.

Similarly, the condition for good 2 to be inferior is

$$J(p,q) = 1 + \frac{pV_p}{qV_q} + \frac{p}{V_q^2}(V_qV_{qp} - V_pV_{qq}) + \frac{p^2}{qV_q^2}(V_qV_{pp} - V_pV_{pq}) < 0\,, \tag{66}$$

which may be converted to the form

$$J(p,q) \equiv \left(1 + \frac{pV_p}{qV_q}\right) H(p,q) - \frac{p^2}{q}C(p,q) < 0\,, \tag{67}$$

with similar interpretation.[7] Since at most one good can be inferior (for any given price pair (p,q)), $V(p,q)$ exhibits inferiority if and only if

$$I(p,q)J(p,q) < 0 \tag{68}$$

anywhere in the positive (p,q) quadrant. Note that, just as for the Giffen pair $\{G(p,q), H(p,q)\}$, $J(p,q) = I(q,p)$ only if $V(p,q)$ is symmetric in p and q. By virtue of the above results, the regions of Giffenity $\mathcal{G}_1(p,q)$ and $\mathcal{G}_2(p,q)$ (if they exist at all) imply the existence of regions of inferiority $\mathcal{I}_1(p,q)$ and $\mathcal{I}_2(p,q)$, with $\mathcal{G}_1(p,q) \subseteq \mathcal{I}_1(p,q)$, and $\mathcal{G}_2(p,q) \subseteq \mathcal{I}_2(p,q)$.

Note further that, just as for the convexity function $C(p,q)$, the functions $I(p,q)$ and $J(p,q)$ are invariant under any functional change of the form $\hat{V}(p,q) = F(V(p,q))$, i.e. they too are properties of the geometry of the contour map of V (but not of the values of V on the contours). This may be easily verified.

4.3 Inferior goods from the DUF $U(x,y)$

The distinction between normality and inferiority can be similarly considered in terms of a direct utility function (DUF) $U(x,y)$. An increase of budget b at constant (p,q) corresponds to parallel displacement of the budget line

$$px + qy = b \tag{69}$$

outwards, away from the origin in the positive (x,y) quadrant. The slope $s(x,y)$ of this budget line where it touches a contour $U(x,y) = \text{const.}$ is given by

$$s(x,y) = -\frac{p}{q} = -\frac{U_x}{U_y}\,, \tag{70}$$

and, under an arbitrary (differential) displacement (dx, dy), the change in s is given by

$$ds = -\frac{U_y dU_x - U_x dU_y}{U_y^2} = \frac{B(x,y)dx - A(x,y)dy}{U_y^2}\,, \tag{71}$$

where A and B are still as defined in (3) above. We choose (dx, dy) such that the slope remains constant, i.e. $ds = 0$, or equivalently

[7] The terms involving $C(p,q)$ in (64) and (67) take the more symmetric form $+(q^2|\nabla V|^3/pV_p^3)\kappa(p,q)$ and $+(p^2|\nabla V|^3/qV_q^3)\kappa(p,q)$, if expressed in terms of the curvature κ (see footnote to eqn.(2)). There are good reasons however for retaining the notation involving the convexity function $C(p,q)$ rather than $\kappa(p,q)$.

$$B(x,y)dx - A(x,y)dy = 0\,. \tag{72}$$

Moreover, if the shift (dx, dy) corresponds to a differential increment db in the budget b, then we have also

$$pdx + qdy = db\,. \tag{73}$$

Hence, solving (72) and (73), we obtain

$$\frac{dx}{db} = \frac{A}{D}\,, \quad \frac{dy}{db} = \frac{B}{D}\,, \tag{74}$$

where

$$D = pA + qB\,. \tag{75}$$

We note that, using (4) and (70),

$$D = \frac{q}{U_y}(U_x A + U_y B) = qU_y^2 C(x,y) \ > \ 0\,, \tag{76}$$

and so also

$$pA + qB \ > \ 0\,. \tag{77}$$

It follows from (74) that good 1 is inferior if and only if $A < 0$ (and $B > U_x|A|/U_y$), and that good 2 is inferior if and only if $B < 0$ (and $A > U_y|B|/U_x$); and that, since at most one of the two goods can be inferior, $U(x,y)$ exhibits inferiority at the point (x,y) if and only if

$$A(x,y)B(x,y) \equiv (U_y U_{xy} - U_x U_{yy})(U_x U_{xy} - U_y U_{xx}) < 0\,. \tag{78}$$

If $A(x,y)B(x,y) > 0$, then both goods are normal. Again, we note that these conditions are invariant under any functional change of DUF of the form $\hat{U}(x,y) = F(U(x,y))$, and are therefore (like the condition (68)) genuine properties of the contour geometry.

It is obvious from (33) that if $G < 0$ then $B < 0$, i.e. if good 2 is a Giffen good, then it is *a fortiori* an inferior good; again, just as for the IUF situation, Giffenity implies inferiority, but not conversely[8].

[8] Of course, the fact that Giffenity implies inferiority can be seen directly from the Slutsky equation: the substitution effect of a price rise is always negative; the income effect is positive for an inferior good, while for a Giffen good the income effect is not merely positive, but *sufficiently* positive to outweigh the substitution effect.

4.4 Examples

Example 1. Let $U(x,y) = yf(x)$, where $f > 0, f' > 0$. Then A and B are given by (10); it follows that good 1 is normal and that, from (8) and (78), good 2 is inferior if and only if

$$f'^2 < ff'' < 2f'^2 \,. \tag{79}$$

For the particular choice $f(x) = x\exp(x^n)$, we have seen that the convexity condition $ff'' < 2f'^2$ is satisfied for $1 \le n \le 5$. Furthermore it is easily verified that the inferiority condition $f'^2 < ff''$ is then satisfied also provided

$$x > \left(\frac{1}{n(n-1)}\right)^{\frac{1}{n}} \,. \tag{80}$$

For example, if $n = 4$, good 2 is inferior for $x > (1/12)^{1/4} \approx 0.537$. This situation is illustrated in figure 7. (Recall that the price ratio $p/q = U_x/U_y$ is determined by the value of x on any chosen contour $U(x,y) = c$.)

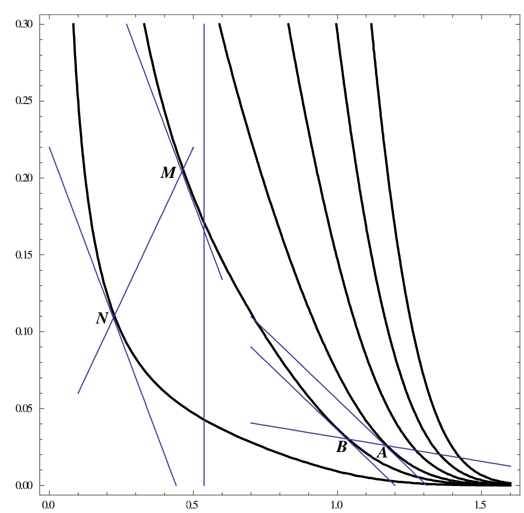

Fig. 7 Contour map for the DUF $U(x,y) = y\,x\exp(x^4)$, showing that good 2 is inferior for $x > 0.537$, where the line BA joining the points of tangency slopes downwards, whereas both goods are normal for $x < 0.537$, where the line NM joining the points of tangency slopes upwards.

Consider now the mirror family of IUFs of the form $V(p,q) = -qf(p)$. Then

$$V_p = -qf'(p), \quad V_q = -f(p), \quad V_{pp} = -qf'', \quad V_{pq} = -f', \quad V_{qq} = 0\,, \tag{81}$$

so that, from (65) and (67), we have after simplification

$$I(p) = \frac{2f'^2 - ff''}{f'^2}, \quad J(p) = \frac{f^2 + 2pff' + p^2(ff'' - f'^2)}{f^2}\,. \tag{82}$$

(The dependence of I and J on q disappears here.) Note that, by virtue of the convexity condition, it is always the case that for this class of IUF,

$$I(p) > 0\,. \tag{83}$$

Also, for the particular mirror choice $f(p) = p\exp(p^n)$, J simplifies to

$$J(p) = 2 + n(n+1)p^n > 0\,. \tag{84}$$

Hence the mirror IUF $V(p,q) = -q\,p\exp(p^n)$ exhibits no inferiority in either good, irrespective of the value of n.

Example 2. As we have shown, any Giffen good is, *a fortiori*, an inferior good. We may then consider our earlier ‘hyperbolic’ $U(x,y)$, as defined by (56), for which good 2 is a Giffen good (and therefore also inferior) in the region $\mathcal{G}_1(x,y)$ (the part of $\mathcal{G}(x,y)$ above the diagonal in figure 5), and good 1 is therefore a Giffen good (and therefore also inferior) in the corresponding region $\mathcal{G}_2(p,q)$ for the mirror IUF $V(p,q)$.

Example 3. Following from Example 1 above, we may ask whether *any* IUF of the form $V(p,q) = -qf(p)$ can exhibit inferiority. One example will suffice: figure 8 shows some contours of the IUF $V(p,q) = -qf(p)$ with

$$f(p) = \frac{1}{1.5 - p + \sqrt{(0.5-p)^2 + 0.001}}\,, \tag{85}$$

for which the contours are all hyperbolic. Three curves (suitably normalised) are superposed on the contour map. The thick dashed curve shows that $J(p)$ as given by (82) is negative in an interval of p around $p = 0.5$ where the contour curvature is pronounced; in this range, good 2 is indeed an inferior good. The thin and thick solid curves show $10\,A$ and B respectively for the mirror DUF $U(x,y) = -V(x,y)$. These functions are evidently positive for $x(\leftrightarrow p)$ in this interval, so that for $U(x,y)$ both goods are normal in the range in which good 2 for $V(p,q)$ is inferior.

These examples demonstrate that, unlike the situation for Giffen goods, no general conclusion may be drawn concerning inferior goods: from example

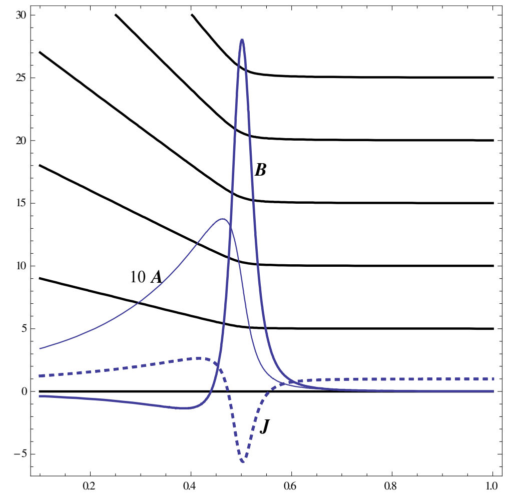

Fig. 8 Contour map of the IUF $V(p,q) = -qf(p)$, with $f(p)$ given by (85). Three curves are superposed on the contours as indicated: $J(p)$ for the IUF, and $A(x)$ (scaled by a factor 10) and $B(x)$ for the mirror DUF. $A(x)$ is positive for all x. $B(x)$ has a range of negativity where good 2 (for the mirror DUF) is inferior. The IUF shows an interval around $p = 0.5$ in which $J(p)$ is negative so that good 2 (for the IUF) is inferior in this interval of p, whereas for the mirror DUF both A and B are positive in the corresponding x-interval so that both goods are normal there.

1, it is evident that inferiority of a good for the DUF $U(x, y)$ provides no guarantee of inferiority of either good for the mirror IUF $V(p, q)$. Example 2 shows that, for some choices of $U(x, y)$ (in particular those for which Giffen behaviour is present), such reciprocal behaviour does occur. Finally, example 3 shows that there exist IUFs that show inferiority in one good, while the mirror DUF shows no inferiority in the corresponding region. Although the general conclusion is negative, the criteria (68) and (78) are nevertheless useful in that they provide immediate tests for inferiority of any given IUF or DUF respectively, and in each case the good that is inferior and the associated domain of inferiority are easily determined.

5 Necessity and luxury goods

5.1 Luxury goods from the DUF

Let us now consider the necessity/luxury dichotomy. A good is a 'luxury' if the *proportion* of income spent on it increases with increasing budget; it is a 'necessity' if this proportion decreases with increasing budget. Obviously in a two-good situation, if one good is a luxury, then the other must be a necessity . If neither good is a luxury, then the proportion of income spent on either good remains constant with increasing budget, the case of homothetic preferences. It is well known that the Cobb-Douglas utility function $U(x, y) = x^{\alpha} y^{\beta}$ (with $\alpha > 0$, $\beta > 0$) satisfies homotheticity: neither good is a luxury for any (x, y).

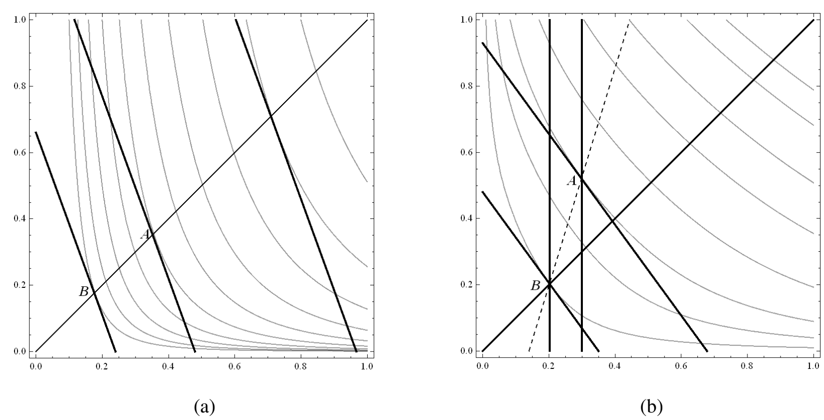

Fig. 9 (a) Contours of the Cobb-Douglas DUF $U_0(x, y) = yx^3$; tangents to contours at points on any ray through the origin are all parallel, indicating homothetic preferences: the increase in x and y between any two points of tangency (e.g. B and A) is proportional to the increase of budget b. (b) Contours of the DUF $U_1(x, y) = yx^2 + xy^3$; here the dashed line BA slopes upwards relative to the radius vector OB, indicating that the increases in x and y are respectively less and greater than proportional to the increase in b; thus in this case, good 1 is a necessity and good 2 a luxury.

The situation may be illustrated with reference to figure 9, which shows the contours of two sample DUFs: (a) $U_0(x, y) = x^3 y$ and (b) $U_1(x, y) = yx^2 + xy^3$. The first of these is homothetic, as evidenced by the fact that the points of tangency of parallel budget lines all lie on the same ray through the origin; this means that the increases of x and y are both proportional to the increase in the budget b. Contrast this with the situation of (b), where the points of tangency veer upwards relative to the ray OB from the origin O, indicating that the increases in x and y are respectively less and greater than proportional to the increase in b: here good 1 is a necessity and good 2 a luxury.

For general $U(x, y)$, with budget constraint $px + qy = b$, the proportion of income spent on good 1 is px/b, so the condition that good 1 be a luxury is

$$L(x, y) \equiv \frac{d}{db}\left(\frac{px}{b}\right) > 0\,. \tag{86}$$

We may describe $L(x, y)$ as the 'luxury function' associated with $U(x, y)$.

From (86), the price p being held constant, we have

$$L(x, y) = p\left[\frac{1}{b}\frac{dx}{db} - \frac{x}{b^2}\right]\,. \tag{87}$$

Obviously, $L > 0$ implies $dx/db > x/b > 0$, i.e. if good 1 is a luxury, it is also normal (in the sense of §4.1 above). Further, with $A(x, y)$ and $B(x, y)$ as defined in (3) above, and using (74) and $px + qy = b$,

$$L(x, y) = \frac{p}{b}\left[\frac{A}{pA + qB} - \frac{x}{b}\right]\,. \tag{88}$$

Recall that, from (75) and (76),

$$pA + qB = D = qU_y^2 C\,, \tag{89}$$

where $C(x, y)$ is the (positive) convexity function. Hence from (88)

$$L(x, y) = \frac{p}{b}\left[\frac{A}{qU_y^2 C} - \frac{x}{b}\right] = \frac{p}{bU_y^2 C(x, y)}(yA(x, y) - xB(x, y))\,. \tag{90}$$

It follows that a necessary and sufficient condition for good 1 to be a luxury at (x, y) is that

$$\hat{L}(x, y) \equiv yA(x, y) - xB(x, y) > 0\,. \tag{91}$$

Note from the first form of $L(x, y)$ in (90) that if good 1 is a luxury then A must be positive (to ensure that L be positive.) However B may be positive or negative (consistent with (89) and (91)); i.e. although good 1 is normal, good 2 (which is in these circumstances a necessity) may be either normal or inferior.

Finally, note that if

$$yA(x, y) - xB(x, y) = 0\,, \tag{92}$$

then both goods are 'neutral' (in that they exhibit homothetic preferences). For Cobb-Douglas utility functions, this equation is satisfied identically for all (x, y). More generally, (92) defines a curve on one side of which good 1 is a necessity and good 2 a luxury, on the other good 1 is a luxury, good 2 a necessity. An example is given below.

5.2 Examples

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Example 1. Consider first the situation when $U(x, y)$ takes the special form considered in §1,

$$U(x, y) = yf(x)\,, \tag{93}$$

with $f > 0, f' > 0$, for which A and B are given by (10). Hence

$$yA - xB = y\left[ff' - x(f'^2 - ff'')\right]\,, \tag{94}$$

and, in this circumstance, from (91), good 1 is a luxury if and only if

$$ff' - x(f'^2 - ff'') > 0\,. \tag{95}$$

For example, if we choose

$$f(x) = x + \alpha x^2\,, \tag{96}$$

where $\alpha > 0$, then $f' = 1 + 2\alpha x$, $f'' = 2\alpha$, and after simplification, we find

$$ff' - x(f'^2 - ff'') = \alpha x^2 \;>\; 0\,, \tag{97}$$

so good 1 is a luxury for all (x, y). Similarly, if we choose

$$f(x) = xe^x\,, \tag{98}$$

then we find

$$ff' - x(f'^2 - ff'') = x^2 e^{2x} \;>\; 0\,, \tag{99}$$

and again good 1 is a luxury for all (x, y).

Example 2. A more interesting example can be constructed as follows. The DUF $U_1(x, y) = yx^2 + xy^3$ has a luxury function that is negative everywhere (in fact $\hat{L}_1(x, y) = -5x^2y^3$), so that for U_1, good 1 is a necessity, as already recognised from figure 9(b). By symmetry, the DUF $U_2(x, y) = xy^2 + yx^3$ has a luxury function that is positive everywhere ($\hat{L}_2(x, y) = 5x^3y^2$), so that for U_2, good 1 is a luxury. Now let

$$U(x, y) = U_1(x, y) + (U_2(x, y))^2 = yx^2 + xy^3 + (xy^2 + yx^3)^2\,. \tag{100}$$

It may be verified that the convexity function for this DUF is positive, so that it is, as required, quasi-concave. It is also the case that both goods are normal for all (x, y). Near the origin, U has approximately the same contours as U_1, while far from the origin U has approximately the same contours as U_2^2, and these are the same as the contours of U_2. Hence for the composite DUF $U(x, y)$, we may expect the luxury function $L(x, y)$ to be negative near,

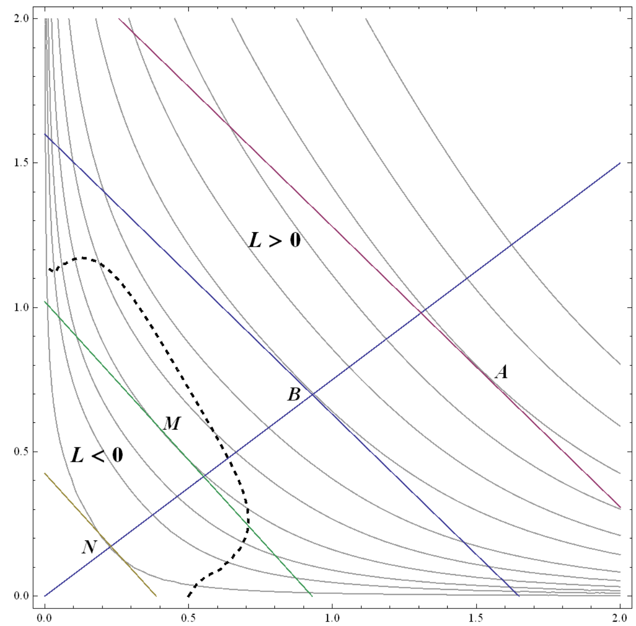

Fig. 10 Contours of the DUF $U(x, y)$ given by (100). The associated luxury function $L(x, y)$ vanishes on the dashed curve, and is negative to the left of it, positive to the right. When the budget increases at fixed prices, the budget line at the point of tangency N moves outwards with constant gradient, and the point of tangency moves in an anticlockwise sense (relative to the radius vector from the origin) towards the position M, indicating a less than proportionate increase in the consumption of good 1, which is therefore a necessity in this region. The situation is the opposite with the movement of the tangent at B out to that at A; in this external region, good 1 is a luxury.

and positive far from, the origin. This is indeed found to be the case. Figure 10 shows contours of the DUF (100), and, superposed, the dashed curve on which $L(x, y) = 0$. On this curve, the preferences are homothetic. To its left $L < 0$; in this region, an increase of budget moves the point of tangency N to M, with a less than proportionate increase in x (cf. figure 9); hence in this region, good 1 is indeed a necessity. In the outer region to the right of the dashed curve, $L > 0$, and outward movement of the budget line moves the point of tangency from B to A, showing a greater than proportionate increase in x, i.e. here good 1 is a luxury. Note that NM and BA both slope upwards: good 1 and good 2 are both normal in both regions.

5.3 Luxury goods from the IUF

We again take the budget constraint in the form

$$px + qy = 1 \tag{101}$$

with the budget normalised to unity. The proportion of the budget spent on good 2 is

$$\frac{qy}{px + qy}\,. \tag{102}$$

We now decrease p and q, with $dp/p = dq/q$, i.e. on the line $q = kp$, with k constant. Good 2 is a luxury (and so good 1 a necessity) if the proportion (102) increases with increasing budget, i.e. if

$$\left[\frac{d}{dp}\left(\frac{px + qy}{qy}\right)\right]_{p/q=const.} > 0. \tag{103}$$

Using Roy's identity,

$$x = \frac{V_p}{pV_p + qV_q}, \quad y = \frac{V_q}{pV_p + qV_q}\,, \tag{104}$$

this leads to the condition

$$\left[\frac{d}{dp}\left(\frac{V_p}{V_q}\right)\right]_{p/q=const.} = \frac{p(V_qV_{pp} - V_pV_{pq}) + q(V_qV_{pq} - V_pV_{qq})}{pV_q^2} > 0\,, \tag{105}$$

which in turn leads to the condition

$$qA(p,q) - pB(p,q) > 0\,. \tag{106}$$

With the correspondences $x \leftrightarrow p, y \leftrightarrow q$, this is identical with the condition (91). We have thus proved a second reflexion theorem as follows:

Theorem 2: Let $U(x,y)$ be a DUF such that good 1 is a luxury/necessity in regions $\mathcal{L}_\pm(x,y)$ respectively of the positive (x,y) quadrant, and let $V(p,q)$ be the mirror IUF defined by $V(p,q) = -U(p,q)$. Then the good 1 associated with $V(p,q)$ is a necessity/luxury respectively in the corresponding regions $\mathcal{L}_\pm(p,q)$ of the positive (p,q) quadrant.

For example, reverting to example 2 in §5.2 above, figure 11 shows the contour map of the mirror IUF

$$V(p,q) = -(qp^2 + pq^3) - (pq^2 + qp^3)^2\,, \tag{107}$$

(the same as figure 10, but drawn with different scaling). The tangents MM', NN' at points M and N to the left of the curve $L = 0$ on a ray through

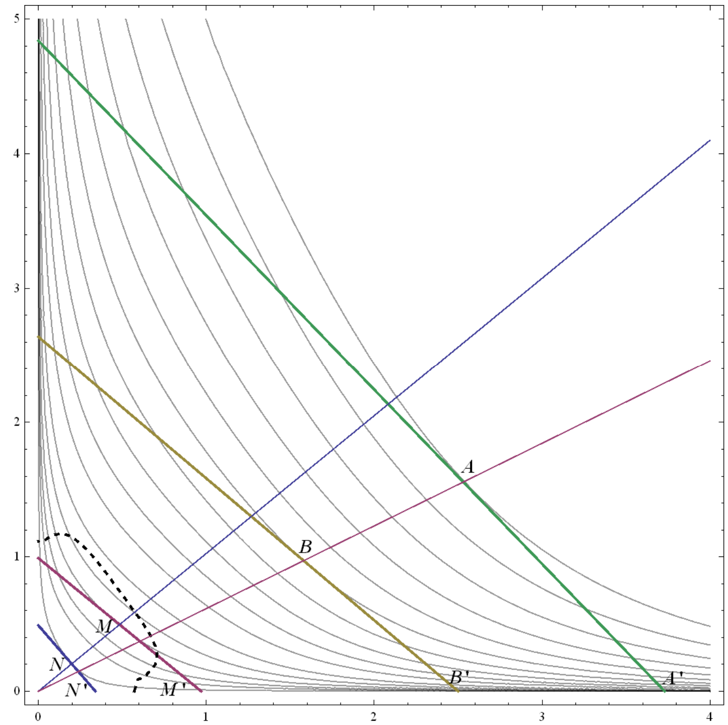

Fig. 11 The mirror IUF $V(p,q)$ given by (107), and drawn to a smaller scale. The tangents MM', NN' at points M and N to the left of the dashed curve $L = 0$ on the upper ray through the origin *diverge* in the downward direction, a symptom of *luxury* for good 1, whereas the tangents AA', BB' at points A and B on the lower ray through the origin *converge* in the downward direction, a symptom of *necessity* for good 1.

the origin *diverge* in the downward direction, a symptom of *luxury* of good 1, whereas the tangents AA' and BB' at points A and B (on a different ray through the origin) *converge* in the downward direction, a weak convergence but nevertheless quite visible, and a symptom of *necessity* for good 1. Thus, as required by theorem 2, good 1 associated with this IUF is indeed a luxury to the left of the dashed curve, and a necessity to its right, just the opposite of the situation for the DUF (100), of which the IUF (107) is the mirror. Note that if the tangent AA' in figure 11 is rolled slightly on its contour in the anti-clockwise sense to become parallel with the tangent BB', then its point of tangency moves *down* the contour, in conformity with figure 10. Likewise, if the tangent MM' is rolled clockwise on its contour to become parallel with NN', then its point of tangency moves *up* the contour, again in conformity with figure 10. This provides an illuminating geometrical interpretation and justification of Theorem 2.

6 Conclusions

Our initial aim in this paper was to explore relations between demand features of a direct utility function $U(x, y)$ and its mirror indirect utility function $V(p, q)$ defined by $V(p, q) = -U(p, q)$. We have shown that both Giffen goods and luxury goods exhibit a 'reflexion' property, whereby, if good 1 is either a Giffen good or a luxury good for the IUF in some region of the (p, q)-quadrant, then good 2 is either a Giffen good or a luxury good, respectively, for the mirror DUF in the corresponding region of the (x, y)-quadrant; and the converse is also true. This reflexion property does not however carry over to the properties of 'inferiority' or 'normality' of goods. There is therefore nothing obvious about the results, although both admit clear geometrical interpretation that makes crucial use of the axiomatic properties of monotonicity and convexity of the contours of either IUF or DUF.

A useful side-product of this analysis is that we have obtained expressions[9] representing 'degree of convexity' $C(\mathbf{x})\,(>\ 0)$, 'degree of Giffenity, $\{G(\mathbf{x}), H(\mathbf{x})\}$, 'degree of inferiority', $\{A(\mathbf{x}), B(\mathbf{x})\}$, and 'degree of luxury/necessity', $L(\mathbf{x})$, of goods 1 and 2 respectively. For any given $U(\mathbf{x})$, these functions can be immediately computed. Here, the sign conventions are such that good 2 is a Giffen good where $G(\mathbf{x}) < 0$, good 1 is a Giffen good where $H(\mathbf{x}) < 0$, good 1 is inferior where $A(\mathbf{x}) < 0$, good 2 is inferior where $B(\mathbf{x}) < 0$, and good 1 is a necessity where $L(\mathbf{x}) < 0$. The domains of negativity of these functions are therefore of particular interest, as are the locations of their minima, where the properties in question are most pronounced.

We have obtained a corresponding set of functions associated with an arbitrary IUF $V(\mathbf{p})$: $C(\mathbf{p})\,(>\ 0), H(\mathbf{p}), G(\mathbf{p}), I(\mathbf{p}), J(\mathbf{p}), L(\mathbf{p})$, with similar interpretations.

Our theory has been concerned with static situations. A useful generalisation would be to consider the dynamic utility function $U(\mathbf{x}, t)$, where t is a time variable. The situation is reminiscent of the dynamics of two-dimensional, incompressible flow, for which $U(\mathbf{x}, t)$ plays the part of a stream function, the contours $U = \text{const.}$ being the instantaneous streamlines of the flow. Such a flow evolves according to the well-known equations of fluid dynamics, which incorporate pressure gradients in conjunction with any force fields that may be present. Similarly, in the economic context, $U(\mathbf{x}, t)$ must evolve in response to economic pressures both internal and external to the particular system considered. If an economic model for the evolution of $U = U(\mathbf{x}, t)$ is known, then the corresponding evolution of all the above functions can be immediately computed, e.g. $G = G(\mathbf{x}, t), L = L(\mathbf{x}, t)$, etc. We may then pose such questions as: under what circumstances can a good that is a luxury in one time period become a necessity in a later time period? Or, how can an inferior good evolve into a normal good with the passage of time (as in the example of Turvey 1980, Chap. 1)? Or, how can Giffen-

[9] Here we use vector notation $\mathbf{x} = (x, y)$, $\mathbf{p} = (p, q)$.

ity appear in a changing economic climate? The results of the present paper provide a platform for the future study of such questions.

It is obviously also desirable to extend the analysis to systems with more than two goods. The papers by Quah (2000, 2003) and Kohler (1985) provide a helpful lead in this direction. We have thought it appropriate here however to focus on the simplest two-good situation, in order to provide a firm basis for future work. A further challenge now is to determine how the theory may be extended to cover the multiple-good situation.

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