Environmental Outcomes in a Model of Mixed Duopoly

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Abstract

We show under general demand and cost conditions that in a mixed duopoly with pollution the government will implement the socially optimal outputs and abatements by a tax-subsidy scheme and keeping the public firm fully public. The scheme requires taxing outputs and subsidizing abatements at different rates, unlike a pollution tax. Our result contradicts some of the recent claims that social optimum is not implementable and privatization is necessary. We also show that when the private firm is partly foreign-owned, the government will adopt some privatization and will not implement the social optimum, though the social optimum is implementable.

Key words: Environmental damage, mixed duopoly, privatization, tax-subsidy scheme, foreign firm

JEL Classifications: H23, Q50, Q58, L13, L33

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1 Introduction

Recently a number of articles have examined whether firm ownership matters for the performance of environmental policies. Using pollution tax as a policy instrument Beladi and Chao (2006), Wang et al. (2009) and Naito and Ogawa (2009) argued that a publicly owned firm tends to pollute more, whether it is alone in the industry (as in Beladi and Chao, 2006) or facing duopoly competition. In particular, these papers show that government cannot achieve the full social optimum; however, it does better by partially privatizing the public firm in conjunction with an optimal pollution tax.[1] This negative result may be seen as an extension of a key feature of the mixed oligopoly literature since deFraja and Delbono (1989). In deFraja and Delbono (1989) it was shown that when one or several private firms co-exist with one public firm the social welfare (ignoring any environmental concerns) will be less than maximum, and, more surprisingly, the social welfare can improve sometimes if the number of private firms increases. The reason is that under diminishing returns technology the social optimum requires distributing the industry output among firms in such a manner that each will produce at a point where price is equal to marginal cost. The private firms do not reach this desirable situation on their own due to their market power. The presence of the public firm in their midst does not make things better, because due to its non-profit maximizing objective the public firm will produce much more (while still maintaining price equal to marginal cost) and in response private firms will produce far less widening the gap between their marginal cost and the market price. Adding more private firms in this setup can force the public firm to cut its output and the distribution of output between firms can improve. Matsumura (1998) built on this insight and showed that generally it is optimal to privatize the public firm partially. Partial privatization in Matsumura (1998) has similar effects as increasing the

[1] Barcena-Ruiz and Garzon (2006) argued that environmental damage is lower in case of private oligopoly than that in case of mixed oligopoly with one fully public firm. Wang and Wang (2009) also argued that, if products are highly substitute, private duopoly pollutes less. These papers did not consider the possibility of partial privatization.

number of private firms in deFraja and Delbono (1989). The subsequent development in the mixed oligopoly literature examined many issues, but notably subsidizing the private firms to achieve the social optimum and below marginal cost pricing of the public firm in the presence of a foreign firm. Recently, environmental issues also have found their way into this literature and the contributions of Beladi and Chao (2006), Wang et al. (2009) and Naito and Ogawa (2009) echo the same negative view of full public ownership. On the other hand, Cato (2008) argues that full public ownership may be socially desirable in some cases.[2]

There are four areas of dissatisfaction of this nascent mixed oligopoly models of environment. First, most of the models have considered specific demand and cost functions, and therefore it is difficult to speculate how these results will stand up to wider set of demand and cost conditions. Second, often abatement is treated as an unrestricted variable, though output seems to be a natural upper bound. Third, the policy space of the government considered is fairly restricted; usually it is confined to pollution tax, which penalizes output and subsidizes pollution abatement (simultaneously) at the same rate. This feature makes pollution tax somewhat special within the class of tax-subsidy based environmental policies. Fourth, in the environmental economics literature a wider set of policies are considered including tradeable permits and pollution standards (see, for example, Jung et al., 1996; Silva and Zhu, 2009; Bhattacharya and Pal, 2010; Colla et al., 2011).[3] We believe that considering the full set of policies will provide much more robust

[2]If the marginal cost of production is decreasing in emission and marginal environmental damage due to pollution is very large, mixed duopoly with a fully public firm leads to higher social welfare than that in case of private duopoly.

[3]We note here that existing literature on optimal environmental policy in the context of oligopolistic industries helps us to understand a variety of other issues: role of product differentiation and free entry (Canton et al., 2008; Fujiwara, 2009), consequences of asymmetric information (Antelo and Loureiro., 2009), implications of strategic managerial delegation (Pal, 2010; Barcena-Ruiz and Garzon, 2002) link between pollution taxes and financial decisions of firms (Damania, 2000), strategic choice of environmental policy in case of open economies (Conrad, 1993; Kennedy, 1994; Barrett, 1994; Ulph, 1996; Bhattacharya and Pal, 2010), so on so forth. The issue of mixed duopoly has not received much attention in this strand

understanding of mixed duopoly, though we do not pursue it here.

We address the first three concerns. We allow the demand and cost functions to be fairly general. At the same time we make the policy space slightly larger, though our attention is restricted to tax and subsidies; to be more specific, we allow the tax rate on output and the rate of subsidy on pollution to be different. Further, we explicitly incorporate the constraint that abatement of a firm should not exceed its output. This restriction seems natural when pollution occurs at the stage of production and abatement takes the form of clean-up; some of the existing models have ignored this constraint.

Our key result is that the social optimum is achieved by taxing the output at a lower rate than the rate at which the abatement is to be subsidized. The tax-subsidy scheme does not directly affect the public firm's output and abatement choices (which are always efficient), but it does induce the private firm to produce and abate at the socially optimum levels if the tax and the subsidy are set appropriately. The tax rate on output can also be negative, if the marginal environmental damage is relatively small. Thus, the negative view of public ownership that some of the authors have stressed is a result of relying on a special tax-subsidy scheme. We also establish that unless there are constant returns in both production and abatement (so that the marginal costs of production and abatement are both constant), it is always optimal to have the private firm produce a strictly positive output. That is, the public firm should not drive out the private firm.

What if the government is free to choose privatization as well? The answer is that the government should not privatize *at all*; with full public ownership and socially optimal tax on output and subsidy on abatement the social optimum is achieved.[4] This is a somewhat stronger assertion running contrary to the inherently critical mixed oligopoly literature. But it is not surprising. If the government can target different determinants of inefficiency by different instruments (i.e. output by tax, abatement by subsidy and market power by

of literature.

[4]This, however, is not the uniquely optimal policy. To se that consider full privatization of the public firm, and then setting appropriate tax on output and subsidy on abatement will also implement the social optimum.

public ownership), then there is no reason why the first best cannot be achieved.

We also consider several special cases. For example, we consider the case of pollution tax, where the tax rate on output and the subsidy rate on abatement are same. It is shown that the government *cannot* achieve the social optimum. If output price is equated to social marginal cost, marginal abatement cost will not be equal to marginal environmental damage, and vice versa. The optimal pollution tax in this environment will force the private firm to abate more than the public firm. In particular, if the slope of the marginal cost (of production) function is small, the private firm will be induced to abate fully – an implication of the abatement constraints which have been ignored in the literature. Thus, we may witness a situation where the private firm completely cleans up its pollution, but the public firm leaves some of its pollution unabated – a situation that makes the public firm more polluting than its private counterpart; but this is perfectly consistent with the policy of social welfare minded government.

Next, considering pollution tax along with privatization we show that optimal pollution tax needs to be complemented by partial privatization, an observation already made in the literature. Wang et al. (2009) and Naito and Ogawa (2009) considered examples with strictly increasing marginal costs of production and abatement and arrived at the partial privatization result, echoing Matsumura (1998)'s partial privatization result in the absence of any externality. We show that marginal cost of production need not be strictly increasing for this result, which is crucial in Matsumura (1998) for the absence of externality; with increasing marginal cost of abatement also one gets the same result. This can be seen as a case of privatization for the sake of environment. A very special case is the case of no externality. There we replicate the Matsumura (1998) result of partial privatization by exogenously setting tax to be zero, and alternatively keeping the public firm fully public we replicate the White (1996)'s result of subsidizing the private firm.

Finally, we consider the case of the private firm being foreign. Here too we show that the first best outputs and abatements can be implemented by an appropriate tax-subsidy scheme and keeping the public firm fully public. However, here arises a paradoxical situation. As long as all firms are domestic taxes and subsidies are essentially redistributed within the economy, the government has no additional preference for tax revenue or aversion to subsidies. But a foreign firm repatriates some (if not all) of its profit. Therefore, tax collected on its repatriated profit is gain to the government and subsidies given to its abatement is partly a leakage. Hence the government develops a preference for tax and aversion to subsidy vis-a-vis the foreign firm. This in turn, will discourage it from implementing the social optimum. In other words, we may have a paradoxical situation. The first best is implementable, but the government may not find in its interest to implement it. In this case the government will opt for some privatization.

The rest of the paper is organized as follows. The next section describes the basic model and characterizes the social optimum, and then shows whether the social optimum is implementable with a simple pollution tax or a more general tax-subsidy scheme. The same question is examined in Section 4 by allowing privatization to be an additional choice variable for the government. Section 4 analyzes the implications of the private firm being foreign-owned. Section 5 concludes. All proofs and examples are placed in the Appendix.

2 The setup

Suppose there is a public firm and a private firm, both engaged in product market competition for a good that involves pollution at the level of production. Let the production cost of firm i be given by an increasing function $C_i = C_i(q_i), C_i'(.) > 0, C_i''(.) \geq 0$, where q_i refers to firm i's output. We consider that firms undertake abatement measures at the time of production to reduce pollution. Abatement taken by firm i is denoted by a_i and the abatement cost is given by the following function: $g_i(a_i), g_i'(.) > 0, g_i''(.) \geq 0$. Abatement is a reduction of pollution arising at the stage of production; hence it is natural to assume that each firm's abatement does not exceed its production level. Pollution of firm i is a linear function of output and abatement; we write it as $h_i = q_i - a_i$. Social damage from aggregate pollution is given by $E = E(q_1 + q_2 - a_1 - a_2), E' > 0, E''(.) \geq 0$. Denoting

$q_1 + q_2$ as Q and $a_1 + a_2$ as A we write $E = E(Q - A)$. The inverse market demand for the good is given as $p = p(Q), p'(.) < 0$. We also assume for simplicity $p''(Q) \leq 0$. Each firm pays an output tax t per unit of q_i and receives a subsidy s per unit of a_i. Note that subsidy instruments, such as subsidy to environmental R&D and other abatement investment projects, soft loans for technology upgrade, depreciation allowance, etc., are often used in practice to abate pollution at the production stage (Lehmann, 2011; Jenkins and Lamech, 1992).[5] A special case of this tax-subsidy regime is pollution tax when $t = s$. In that case, each firm's net payment is $T_i = th_i = t(q_i - a_i)$. Since $a_i \leq q_i$, the payment T_i is strictly nonnegative. But we allow $t \neq s$, and more importantly, while $s \geq 0$ (i.e. s is always a subsidy) t is not restricted to be positive; thus permitting it to be negative we allow t to be subsidy as well.

Profit of firm i is given by

$$\pi_i = p(Q)q_i - C_i(q_i) - g_i(a_i) - tq_i + sa_i, \quad i = 1, 2.$$

Social welfare arising from an aggregate output Q is given by the total benefits (or utility) from the consumption of Q minus the total social cost of producing it. This turns out to be the sum of consumer surplus and industry profit minus social damage as shown below.[6]

$$W = \int_0^Q p(x)dx - C_1(q_1) - C_2(q_2) - g_1(a_1) - g_2(a_2) - E(Q - A). \tag{1}$$

We assume that $W(.)$ is strictly concave in (q_1, q_2, a_1, a_2). This is to ensure that both outputs and abatments are socially desirable goods, and there is a unique maximum. Firm 2's profit function $\pi_2(q_2, a_2)$ is concave in (q_2, a_2). A similar assumption will apply to firm 1's profit wherever relevant.

[5]For example, in China environmental tax-subsidy scheme is such that up to 80 percent of the revenue is spent to subsidize firm-level pollution abatement projects (Wang and Chen, 1999). Similar mechanism is in place in many other countries around the world, including developing and transition economies (Lehmann, 2011; Lovei, 1995).

[6]The industry profit should not take into account any tax or subsidy.

Social optimum: We first determine the socially optimal outputs and abatements, which is derived by maximizing social welfare in (1) with respect to q_i and a_i (ignoring the abatement constraints) as

$$p(.) - C_i'(q_i) - E'(.) = 0, \quad i = 1, 2, \tag{2}$$

$$-g_i'(a_i) + E'(.) = 0, \quad i = 1, 2. \tag{3}$$

Definition 1. ***(Social optimum)*** (q_i^S, a_i^S), $i = 1, 2$, *which solve Eqs.* (2)-(3) *is the socially optimal output and abatement of firm i.*

We assume $q_i^S > a_i^S$ for $i = 1, 2$. In the social optimum, price is equal to the social marginal cost, and each firm's abatement is such that marginal cost of abatement is equal to marginal environmental damage. In the case of symmetric technologies, firms should produce the same output and undertake same abatement.[7]

3 Optimal tax-subsidy scheme and privatization

We would like to see whether the socially optimal outputs and abatements can be implemented by an appropriate choice of tax and subsidy. For this purpose we propose a two-stage mixed duopoly game, in which the first stage concern's the government's choice of tax and subsidy; in the second stage both firms simultaneously choose their outputs and abatements subject to their respective abatement constraints: $a_i \leq q_i$. The firms' choices are influenced by the tax and subsidy applicable to them, and also the objectives they are pursuing.

We assume that firm 2 is fully private, and so it is interested in maximizing profit, π_2. In contrast , firm 1 may be fully or partly owned by government. Let the degree of

[7]We implicitly assume that $q_1 > 0, q_2 > 0$. This requires assuming that the cost asymmetry is not too large. More formally, we need at $q_1 = 0$ $p'(q_2^S) > C_1'(0) + E'(q_2^S - a_2^S)$ and at $q_2 = 0$ $p'(q_1^S) > C_2'(0) + E'(q_1^S - a_1^S)$.

private ownership in firm 1 be denoted by $\theta \in [0, 1]$. Consequently, the objective function of the partially privatized firm is given by a weighted average of its profit (π_1) and the government's objective, which turns out to be social welfare. The objective function of firm 1 is given by $O = \theta\pi_1 + (1-\theta)W$.[8]

We should note that the government's objective may not necessarily be to maximize the social welfare of the economy, because it may have tax preference. However, generally taxes and subsidies are redistributed within the economy, and therefore the government's objective, which we denote as G, becomes identical to social welfare, W. To see this, let G be given the sum of consumer surplus, net industry profit (after tax and subsidy adjustments) and tax revenues, minus the social damage and subsidies given. Thus,

$$\begin{aligned} G &= \left[\int_0^Q p(x)dx - pQ\right] + [pQ - C_1(q_1) - C_2(q_2) - g_1(a_1) - g_2(a_2) - tQ + sA] \\ &\quad - E(Q-A) + tQ - sA \\ &= \int_0^Q p(x)dx - C_1(q_1) - C_2(q_2) - g_1(a_1) - g_2(a_2) - E(Q-A) = W. \end{aligned}$$

Consider the second stage output and abatement choices. Firm 1 maximizes O with respect to q_1 and a_1, while firm 2 maximizes π_2 with respect to q_2 and a_2.[9] The first order conditions for maximization are:

$$\frac{\partial O}{\partial q_1} = p(.) - C_1'(q_1) - (1-\theta)E'(.) + \theta q_1 p'(.) - \theta t = 0 \tag{4}$$
$$\frac{\partial O}{\partial a_1} = -g_1'(a_1) + (1-\theta)E'(.) + \theta s = 0, \tag{5}$$
$$\frac{\partial \pi_2}{\partial q_2} = p(.) + q_2 p'(.) - C_2'(q_2) - t = 0 \tag{6}$$
$$\frac{\partial \pi_2}{\partial a_2} = -g_2'(a_2) + s = 0 \tag{7}$$

[8] It is commonly assumed that the level of privatization (θ) determines the bargaining power of the private partner in bargaining over the payoff with the public sector. Alternatively, following Fershtman (1990), if we consider that the private partner and the public sector bargain over the quantity of output to be produced, where bargaining powers are determined by respective share holdings, qualitative results of this analysis go through. The reason is the formulations of Fershtman (1990) and Matsumura (1998) lead to comparable objective functions of the partially privatized firm (Kumar and Saha, 2008; Saha, 2009)

[9] We assume that $q_1 \geq a_1$ and $q_2 \geq a2$ in the solution.

The Nash equilibrium outputs and abatements $(q_1^*(t,s), a_1^*(t,s)), (q_2^*(t,s), a_2^*(t,s))$ are determined from Eqs. (4)-(7), assuming that the second order conditions hold.[10] Given the assumptions on the primitives of the model, we can assume that the equilibrium will be unique. See Appendix for a more formal argument of the uniqueness.

Before characterizing the optimal policy it would be instructive to make a key observation, which has been ignored in the mixed duopoly literature, though it has appeared in several papers such as Wang et al. (2009) and Naito and Ogawa (2009) under special cases.[11] This observation concerns the role of the private sector.

Proposition 1. ***(Positive output of the private firm)*** *Suppose firms have identical technologies: $C_1(.) = C_2(.) = C(.)$ and $g_1(.) = g_2(.) = g(.)$. Unless $C''(q_i) = 0$ and $g''(a_i) = 0$ for $i = 1, 2$, the optimal tax and subsidy would be such that the private firm produces a strictly positive output at all $\theta \in [0, 1]$.*

Proposition 1 says that in general it will not be optimal to force the private firm to choose zero output (through tax and subsidy), assuming that technologically firms are similar. Ordinarily, if the marginal cost of production and abatement are both constant, social welfare is maximized by concentrating all of the production abatement in a 'fully' public firm, because the fully public firm equates the marginal returns with its actions with the social marginal costs. But if the marginal cost of abatement (and/or production) is strictly increasing, concentrating all of the abatement (and/or production) in a single firm will not be optimal. If the private firm is forced to choose zero output, its abatement will also be zero. Hence, if not for the sake of production, for the sake of abatement production in the private firm must be strictly positive. Thus, even if the marginal cost of

[10] We assume that the second order conditions for each firm's optimization hold, and in addition the stability condition of the Nash equilibrium holds.

[11] Both of these papers have considered specific examples with strictly increasing marginal cost of production and abatement; further the abatement cost function is identical to the output cost function. But our general formulation shows that neither of these assumptions is necessary.

production is constant, production may need to be maintained in both firms for the sake of environment.

Now we consider the optimal tax-subsidy policy and the degree of privatization in firm 1. By the very premise of our model, the government can penalize the polluting outputs and reward the abatements at different rates, and in addition it can optimally privatize the public firm. This array of policies allows implementation of the Pareto optimal output and abatements. Formally, $(q_1^*(t,s), a_1^*(t,s)), (q_2^*(t,s), a_2^*(t,s))$ is substituted in the social welfare function $W(.)$ as given in Eq. (1). The government chooses (θ, t, s) such that Eqs. (4)-(7) becomes identical to Eqs. (2)-(3) for each i.

Proposition 2. ***(Optimal policy)*** *The government will choose zero privatization (i.e. $\theta = 0$) and in addition set $t^* = E'(q_1^S + q_2^S - a_1^2 - a_2^S) + q_2^S p'(q_1^S + q_2^S)$ and $s^* = E'(q_1^S + q_2^S - a_1^2 - a_2^S)$ in order to implement the social optimum.*

Corollary 1. *If firms are identical in terms of their production and abatement technologies, the degree of privatization does not matter. Social optimum is implementable via the tax-subsidy scheme specified in Proposition 2 for any given $\theta \in [0,1]$.*

Proposition 2 says that for the social optimum, the government will keep firm 1 in full public ownership, and induce it to choose the socially optimal output and abatement via social welfare maximization. Simultaneously, it will induce the private firm to produce and abate at the socially optimal level through a tax-subsidy scheme. Note that the government will subsidize the abatements at a higher rate than it will penalize the output; this is clear from the fact that $t^* = E'(q_1^S + q_2^S - a_1^2 - a_2^S) + q_2^S p'(q_1^S + q_2^S) < s^* = E'(q_1^S + q_2^S - a_1^S - a_2^S)$ as $p'(.) < 0$. It is a reflection of the well-known fact that the social optimum is achievable by internalizing all the distortions (imperfect competition and pollution damage in the present context), when the number of policy instruments (production tax and abatement subsidy) equals that of market distortions.

However, one special case arises, as Corollary 1 states, when both firms are identical leading to identical socially optimal output and abatement, i.e. $q_1^S = q_2^S$ and $a_1^S = a_2^S$. In that case one can check from Eqs. (4) and (5) that the proposed tax and subsidy will also make q_1 and a_1 socially optimal, even if $\theta \neq 0$. For example, firm 1 can be fully privatized, and then both firms can be symmetrically treated with the same tax and subsidy rates. But in general, when technology asymmetry is present, zero privatization is necessary to achieve the social optimum.

The zero privatization result appears in sharp contrast to the prescription of partial privatization which figured in several articles. These papers achieve partial privatization because the tax rate on output and the subsidy rate on abatement are the same, as any pollution tax would automatically require so. We formally establish this in the next section.

An important point to note that t^* can be negative implying a potential case of subsidy. Intuitively, if $E'(.)$ is small (i.e. the environmental damage is not significant) then social optimality requires removing the output distortions caused by market power. In that case, the private firm needs to be induced to produced more (and in turn the public firm will respond by cutting down its production) and hence a subsidy would be called for on the output. For abatement, however, a positive subsidy is needed.

Before we conclude this section, it would be worthwhile to examine the possibility of a firm-specific tax-subsidy scheme; after all public firms often in reality receive differential (and favorable) treatments than their private rivals. In a canonical setting like ours firm-specific tax-subsidy schemes may be needed, if the government could not optimally choose θ.[12] If θ is exogenously given (as strictly positive), then by comparing Eqs. (4) and (6) we see that for the social optimum t_1 (i.e. tax on q_1) should be $E'(.) + q_1^S p'(.)$, while t_2 (i.e. tax applied to q_2) should be as specified in Proposition 2. Since generally $q_1^S \neq q_2^S$ (due to technology asymmetry) we have a situation of $t_1 \neq t_2$ required for the social optimum. However, differential tax rates may be politically difficut, and therefore nationalization is

[12] It is not unreasonable to think that privatization may be dealth with a separate divestment authority, completely independent of the ministries of environment or taxation. Besides, privatization decisions may take time due to legal ramifications, and for environmental redresses the government may be able to use only the tax instruments in the short run.

one way of overcoming this difficulty.

4 The case of pollution tax

In this section we consider the special case of pollution tax (where $t = s$ by assumption), and show that the social optimum is not achievable. Now we establish a policy tradeoff intrinsic to pollution tax. The government cannot implement both the socially optimal output and the socially optimal abatement at the same time. There are two sources of inefficiency – market power and negative externality, but the way they inflict inefficiency on the output is different from the way they inflict inefficiency on the abatement. If abatement was not feasible, social optimum would concern only the outputs, and a combination of privatization and pollution tax would ensure the social optimum. But with abatement, that is not the case. While, firm 1's market power can be removed by keeping it fully under public ownership, the private firm's market power and externalities cannot both be corrected by the pollution tax alone. Thus, there is a tradeoff for the social planner between inducing the 'right level of output' and the 'right level of abatement'. One might argue that with technological advancements making abatement and direct offset possible, social optimum may become harder to achieve, because the government may need more sophisticated tax instruments.[13]

We formally establish this by explicitly considering the firms' optimization problems along with the abatement constraints. Denoting the Lagrange multiplier of firm i by λ_i and differentiating the Lagrange functions, $\mathcal{L}_1 = W + \lambda_1(q_1 - a_1)$ and $\mathcal{L}_2 = \pi_2 + \lambda_2(q_2 - a_2)$, corresponding to the optimization problems of the two firms, we arrive at the following

[13]However, with technological advancements social optimum itself changes, and maximum social welfare also rises.

first order conditions:

$$\frac{\partial \mathcal{L}_1}{\partial q_1} = \frac{\partial O}{\partial q_1} + \lambda_1 = p(.) - C_1'(q_1) - (1-\theta)E'(.) + \theta q_1 p'(.) - \theta t + \lambda_1 = 0 \tag{8}$$

$$\frac{\partial \mathcal{L}_1}{\partial a_1} = \frac{\partial O}{\partial a_1} - \lambda_1 = -g_1'(a_1) + (1-\theta)E'(.) + \theta t - \lambda_1 = 0, \tag{9}$$

$$\frac{\partial \mathcal{L}_1}{\partial \lambda_1} = [q_1 - a_1] = 0, \quad \text{or} \quad \lambda_1 = 0 \tag{10}$$

$$\frac{\partial \mathcal{L}_2}{\partial q_2} = \frac{\partial \pi_2}{\partial q_2} + \lambda_2 = p(.) + q_2 p'(.) - C_2'(q_2) - t + \lambda_2 = 0 \tag{11}$$

$$\frac{\partial \mathcal{L}_2}{\partial a_2} = \frac{\partial \pi_2}{\partial a_2} - \lambda_2 = -g_2'(a_2) + t - \lambda_2 = 0 \tag{12}$$

$$\frac{\partial \mathcal{L}_2}{\partial \lambda_2} = [q_2 - a_2] = 0, \quad \text{or} \quad \lambda_2 = 0. \tag{13}$$

The Nash equilibrium outputs and abatements $(q_1^*(t), a_1^*(t)), (q_2^*(t), a_2^*(t))$ are determined from equations (8)-(13), assuming that the second order conditions hold.[14] Given the assumptions on the primitives of the model, we can assume that the equilibrium will be unique.

Note that, irrespective of whether one, both or none of the abatement constraints bind, we will have the following relationship between outputs and abatements:

$$p(.) - C_1'(q_1) - g_1'(a_1) + \theta q_1 p'(.) = 0 \tag{14}$$

$$p(.) + q_2 p'(.) - C_2'(q_2) - g_2'(a_2) = 0. \tag{15}$$

Compare the social optimality conditions (2)-(3) for (q_i, a_i) with Eqs. (14) and (15). For firm 2, by some combination of θ and t we can have either $g_2'(a_2) = E'(.)$, i.e. the social optimality of a_2 alone, or $g_2'(a_2) - q_2 p'(.) = E'(.)$ which yields $p(.) = C_2'(.) + E'(.)$, social optimality of q_2 alone. But we cannot have the social optimality of both q_2 and a_2 at the same time. This observation does not depend on the value of θ or cost asymmetry between the two firms. Similarly, if by some combination of θ and t it is ensured that $g_1'(a_1) = E'(.)$, which gives socially optimal a_1, then we can not have $p(.) - C_1'(.) - E'(.) = 0$ unless $\theta = 0$.

[14]We assume that the second order conditions for each firm's optimization hold, and in addition the stability condition of the Nash equilibrium holds.

That is, we can not have both q_1 and a_1 at the social optimal level, unless firm 1 is fully nationalized.

Proposition 3. ***(Pollution tax and social optimum)*** *There is no (θ, t) that induces the social optimum.*

Second best scenario. Given that the social optimum is not achievable through a pollution tax, the environmental variants of the mixed oligopoly models have focused on the second best scenarios and delved into the question of optimal pollution tax and privatization. But most of these studies (cited earlier) are typically based on examples and they also ignored the abatement constraints, which are not particularly helpful to understand the general case. The answers to the questions of optimal privatization and pollution tax may vary depending on whether the abatement constraints bind or not, and also on how the public firm's output and abatement choice will react to the tax rate set by the government. A comprehensive treatment of these issues is beyond the scope of this paper; we restrict our attention only to a few issues that are relevant to our central analysis. As we try to present an in-depth analysis we organize it in two parts. We show first when public ownership of full, marginal social damage is contained within a bound. In the second part we show that the optimal privatization will be partial, neither full nor zero. Some of the existing results in the literature can be retrieved as special cases from our model. We then also study several examples where abatement constraints are better studied, with explicit solutions derived and comparative statics cleanly established. These examples are relegated to Appendix B.

4.1 Full public ownership and optimal tax

In this subsection we set $\theta = 0$ in Eqs. (8)-(13) to consider the special case of full public ownership. There are several cases to consider depending on which abatement constraint binds.

Case 1 (no constraints bind): Set $\lambda_i = 0$ $(i = 1, 2)$, and from (12) we first see that a_2 increases with t; for the remaining three variables we need to totally differentiate (8), (9),(11) and (12) simultaneously and after solving the resultant simultaneous equations we confirm the following signs (details are provided in Appendix):

$$\begin{aligned} \frac{\partial q_1^*}{\partial t} &> 0, \quad \frac{\partial q_2^*}{\partial t} < 0 \\ \frac{\partial a_1^*}{\partial t} &< 0, \frac{\partial a_2^*}{\partial t} = \frac{1}{g''(a_2)} > 0. \end{aligned} \tag{16}$$

Lemma 1. *Suppose the abatement constraint of neither firm binds. Then with an increase in the pollution tax rate the output of the public firm rises and its abatement falls, while the output of the private firm falls and its abatement rises. Thus, a higher tax rate encourages (discourages) the public (private) firm to pollute more.*

We see that if the pollution tax rises the private firm responds by cutting down its output and raising its abatement, but the public firm does exactly opposite. Since the two firms respond to a tax rise in opposite ways, a natural question to ask: What is the net effect on environmental damage? Clearly the public firm's pollution rises, and the private firm's pollution falls. But does the increase in public firm's pollution crowd out the decrease in the private firm's pollution? For this we need to determine the sign of the following derivative:

$$\frac{\partial E}{\partial t} = E'(.) \left(\frac{\partial q_1^*}{\partial t} + \frac{\partial q_2^*}{\partial t} - \frac{\partial a_1^*}{\partial t} - \frac{\partial a_2^*}{\partial t} \right)$$

It turns out that under general conditions, the sign remains ambiguous, although as a regularity condition we wish this sign to be negative. With some restrictions on the demand and cost conditions we may be able to ascertain the sign, which we show in our Examples 1 and 2, in Appendix B.

Case 2 (both constraints bind): Here set $\lambda_i > 0$ and $a_i = q_i$ for $i = 1, 2$. Then q_1

and q_2 are determined from the following two equations:

$$p(.) - C_1'(q_1) - g_1'(q_1) = 0 \tag{17}$$

$$p(.) + q_2 p'(.) - C_2'(q_2) - g_2'(q_2) = 0. \tag{18}$$

Here two things are noteworthy. First, neither q_1 (and in turn a_1) nor q_2 (and in turn a_2) will be sensitive to t. Second, abatement levels are higher (or no smaller) in this case than those in case 1 (otherwise case 1 would not have given the unconstrained solution). Hence, $g'(q_i) \geq g'(a_i)$. Therefore, (q_1, q_2) satisfying (17) and (18) are each smaller (or no greater) than (q_1, q_2) in case 1.

Case 3 (constraint in firm 1 binds): Now set $\lambda_1 > 0$ and $a_1 = q_1$, but set $\lambda_2 = 0$ and $a_2 < q_2$. Clearly, a_2 depends on t, and in turn q_1 also depends on t. The relationship between q_1 and t will be positive as in Case 1. This can be verified.

Now, we know from Lemma 1 that if the constraints did not bind, we would have $q_1'(t) > 0, a_1'(t) < 0$ and $q_2'(t) < 0, a_2'(t) > 0$. So $q_1(t)$ can intersect $a_1(t)$ only once. So if $q_1 = a_1$ at $t = t_1$, the imposition of the abatement constraint dictates that at all $t < t_1$ also we must have $q_1 = a_1$. Similarly, if $q_1 > a_1$ at $t = t_1$, then clearly $q_1 > a_1$ at all $t > t_1$. Therefore, in this case, with an increase in the pollution tax rate the public firm's output and abatement will both rise, while the output of the private firm falls and its abatement rises. Further, if the constraint binds at $t = t_1$, then at all $t < t_1$ also the constraint will bind, and if the constraint does not bind at $t = t_1$, then at all $t > t_1$ also the constraint will not bind.

Case 4 (constraint in firm 2 binds): Finally set $\lambda_2 > 0$ and $a_2 = q_2$, but set $\lambda_1 = 0$ and $a_1 < q_1$. Now a_2 (or q_2) does not depend on t, and therefore q_1 also does not depend on t. That is, pollution tax has no effect on any of the firms' abatement or output. Moreover, if the constraint binds at $t = t_2$, then at all $t > t_2$ it will also bind, and if the constraint does not bind at $t = t_2$, then at all $t < t_2$ also the constraint will not bind, since $q_2(t)$ can intersect $a_2(t)$ only once.

Note that social welfare does not directly depend on the tax rate (recall (1)). Also, note that firm 1's output or abatement does not directly depend on t either (refer to (8) and (9)); but firm 2's output and abatement do depend on t. Therefore, the effect on social welfare must be channelized through q_2 and a_2, and then indirectly through q_1 and a_1. Also, when the abatement constraint of firm 2 binds, tax rate ceases to have any incremental effect on the outputs or abatements and thereby on social welfare. So the optimal tax will not be chosen in a region where the abatement constraint of firm 2 is already binding. That is, if t^* is optimal, then there cannot be any $t < t^*$ such that $q_2(t) = a_2(t)$. This then leaves with the two possibilities of optimal tax t^*: either $q_2(t^*) = a_2(t^*)$ with $q_2(t) > a_2(t)$ for all $t < t^*$, or $q_2(t^*) > a_2(t^*)$ (i.e. the constraint not binding).

Lemma 2. *Suppose the abatement constraint of firm 1 binds, but that of firm 2 does not. Then with an increase in the pollution tax rate the public firm's output and abatement will both rise, while the output of the private firm falls and its abatement rises. Further, if the constraint binds at $t = t_1$, then at all $t < t_1$ also the constraint will bind, and if the constraint does not bind at $t = t_1$, then at all $t > t_1$ also the constraint will not bind.*

Lemma 3. *If the private firm's (i.e. firm 2's) abatement constraint binds, pollution tax has no effect on any of the firms' abatement or output. Further, if the constraint binds at $t = t_2$, then at all $t > t_2$ it will also bind, and if the constraint does not bind at $t = t_2$, then at all $t < t_2$ also the constraint will not bind.*

Lemmas 2 and 3 are straight forward implications of the uniqueness and monotonicity properties of the functions $q_i(t)$ and $a_i(t)$. Consider Lemma 2. We know from Lemma 1 that if the constraints did not bind, we would have $q_1'(t) > 0, a_1'(t) < 0$ and $q_2'(t) < 0, a_2'(t) > 0$. So $q_1(t)$ can intersect $a_1(t)$ only once. So if $q_1 = a_1$ at $t = t_1$, the imposition of the abatement constraint dictates that at all $t < t_1$ also we must have $q_1 = a_1$. Similarly, if $q_1 > a_1$ at $t = t_1$, then clearly $q_1 > a_1$ at all $t > t_1$. This establishes Lemma 2. The first part of Lemma 3 is a direct implication of (17) and (18) and the second part follows from the fact that $q_2(t)$ can intersect $a_2(t)$ only once.

Optimal tax. Next, consider the issue of optimal tax. We should first note that social welfare does not directly depend on the tax rate (recall (1)). Also note that firm 1's output or abatement does not directly depend on t either (refer to (8) and (9)); but firm 2's output and abatement do depend on t. Therefore, the effect on social welfare must be channelized through q_2 and a_2, and then indirectly through q_1 and a_1. Given our observation made in Lemma 3 that when the abatement constraint of firm 2 binds, tax rate ceases to have any incremental effect on the outputs or abatements and thereby on social welfare. So the optimal tax will not be chosen in a region where the abatement constraint of firm 2 is already binding. That is, if t^* is optimal, then there cannot be any $t < t^*$ such that $q_2(t) = a_2(t)$. This then leaves with the two possibilities of optimal tax t^*: either $q_2(t^*) = a_2(t^*)$ with $q_2(t) > a_2(t)$ for all $t < t^*$, or $q_2(t^*) > a_2(t^*)$ (i.e. the constraint not binding).

Optimal pollution tax will distort both abatement and output of firm 2 from the social optimum. To see this consider the unconstrained case (case 1) and the government's choice of t which results from maximizing W when firms' responses in terms of output and abatement as given in (16) are perfectly anticipated (after setting $\lambda_i = 0, i = 1, 2$).

$$\frac{\partial W(.)}{\partial t} = \frac{\partial W}{\partial q_1}\frac{\partial q_1^*}{\partial t} + \frac{\partial W}{\partial a_1}\frac{\partial a_1^*}{\partial t} + \frac{\partial W}{\partial q_2}\frac{\partial q_2^*}{\partial t} + \frac{\partial W}{\partial a_2}\frac{\partial a_2^*}{\partial t}$$

Using the facts that $\frac{\partial W}{\partial q_1} = \frac{\partial W}{\partial a_1} = 0$ and

$$\frac{\partial W}{\partial q_2} = p(.) - C_2'(q_2) - E'(.), \qquad \frac{\partial W}{\partial a_2} = -g_2'(a_2) + E'(.)$$

and substituting $C'(q_2) = p(.) - q_2 p'(.) - t$ and $g_2'(a_2) = t$ from the first order conditions (11) and (12) we arrive at

$$\Big[t - q_2 p'(.) - E'(.)\Big]\frac{\partial q_2^*}{\partial t} + \Big[E'(.) - t\Big]\frac{\partial a_2^*}{\partial t} = 0. \tag{19}$$

Since $\frac{\partial q_2^*}{\partial t} < 0$ and $\frac{\partial a_2^*}{\partial t} > 0$ the socially optimal tax rate must be such that either $t = E'(.)$ with $q_2 = 0$ or $t < E'(.) < t - q_2 p'(.)$. But we know from Proposition 1 that inducing

$q_2 = 0$ will not be optimal (in general). In the second case, however, the public firm is allowed to operate by setting the tax rate at a level below the social marginal damage.

Proposition 4. *When one firm is fully public in a mixed duopoly, the optimal pollution tax, t^*, will be such that it will induce $0 < a_2 \leq q_2$ and restrict the social marginal damage within the following bounds:*

$$t^* < E'(.) < t^* - q_2 p'(.). \tag{20}$$

The output and abatement of firm 2 will both fall short of the socially optimal level.

Comparative statics: We briefly consider comparative statics on the optimal tax t^*. Let k be a parameter in the social welfare function, such that $\frac{\partial^2 W}{\partial t \partial k} \neq 0$. Then the effect of k on optimal tax can be determined from the following:

$$\frac{\partial t^*}{\partial k} = -\frac{\partial^2 W / \partial t \partial k}{W''(t)}. \tag{21}$$

Since $W''(t) < 0$,

$$\text{sign } \frac{\partial t^*}{\partial k} = \text{sign } \frac{\partial^2 W}{\partial t \partial k}.$$

For example, if k relates to market size we would expect the above sign to be positive. Alternatively, if k relates to cost functions the above sign is expected to be negative, as is illustrated by Example 2 in Appendix B. Example 2 also shows that depending on the value of k different abatement constraints may bind.

4.2 Partial privatization and optimal tax

Now we return to the question of the whole set of optimal policies – both the optimal pollution tax and the optimal pollution. For the sake of simplicity, we consider only the case where the abatement constraints do not bind, and establish how the second stage output and abatement choices vary with respect to θ and t. Setting $\lambda_i = 0$ $(i = 1, 2)$, from (12) we first see that a_2 increases with t. For the remaining three variables we need to totally differentiate (8), (9), (11) and (12) simultaneously and after solving the resultant simultaneous equations we obtain certain relationships, which are summarized in the following lemma (details are provided in Appendix).

Lemma 4. *On the second stage outputs and abatements the pollution tax and privatization have the following effects:*

(a)

$$\frac{\partial q_1^*}{\partial t} \begin{cases} > 0 & \text{if } 0 \leq \theta < \theta_1 \\ = 0 & \text{if } \theta = \theta_1 \\ < 0 & \text{if } \theta_1 < \theta \leq 1 \end{cases} \tag{22}$$

$$\frac{\partial a_1^*}{\partial t} \begin{cases} < 0 & \text{if } 0 \leq \theta < \theta_2 \\ = 0 & \text{if } \theta = \theta_2 \\ > 0 & \text{if } \theta_2 < \theta \leq 1, \end{cases} \tag{23}$$

$$\frac{\partial q_2^*}{\partial t} < 0, \quad \frac{\partial a_2^*}{\partial t} > 0 \quad \forall \theta. \tag{24}$$

where $0 < \theta_1, \theta_2 < 1$. *Thus, pollution of the partially public firm (i.e. firm 1) is rising in* t *if* $\theta \leq \min[\theta_1, \theta_2]$, *and is falling in* t *if* $\theta \geq \max[\theta_1, \theta_2]$. *Pollution of the private firm is always declining in* t.

and (b)

$$\frac{\partial q_1^*}{\partial \theta} < 0, \quad \frac{\partial q_2^*}{\partial \theta} > 0,$$
$$\frac{\partial a_1^*}{\partial \theta} < 0, \quad \frac{\partial a_2^*}{\partial \theta} = 0.$$

Lemma 4 shows that the public firm's response to the pollution tax depends on the degree of public ownership. If it is fully or mostly government-owned, its output and abatement responses will be exactly the opposite of the private firm's as seen in the previous subsection. But if the public firm is sufficiently divested, then it will respond to the tax exactly the same way the private firm does.

In contrast to Lemma 1, none of the firms' output and abatement will be sensitive to tax rate, if both firms' abatement constraints bind. The same is true also in case only the private firm's abatement constraint is binding. On the other hand, if the abatement constraint of the public firm binds, but that of the private firm does not, with an increase in the pollution tax rate the public firm's output and abatement will both rise, while the output of the private firm falls and its abatement rises. See Appendix for details.

Now we try to determine the socially optimal privatization and pollution tax. These are given by the following implicit equations:

$$\frac{\partial W(.)}{\partial t} = \frac{\partial W}{\partial q_1}\frac{\partial q_1^*}{\partial t} + \frac{\partial W}{\partial a_1}\frac{\partial a_1^*}{\partial t} + \frac{\partial W}{\partial q_2}\frac{\partial q_2^*}{\partial t} + \frac{\partial W}{\partial a_2}\frac{\partial a_2^*}{\partial t} = 0, \tag{25}$$

$$\frac{\partial W(.)}{\partial \theta} = \frac{\partial W}{\partial q_1}\frac{\partial q_1^*}{\partial \theta} + \frac{\partial W}{\partial a_1}\frac{\partial a_1^*}{\partial \theta} + \frac{\partial W}{\partial q_2}\frac{\partial q_2^*}{\partial \theta} + \frac{\partial W}{\partial a_2}\frac{\partial a_2^*}{\partial \theta} = 0. \tag{26}$$

Proposition 5. *The government will set a tax rate t^* no greater than the marginal environmental damage $E'(.)$ and partially privatize the public firm. That is, the optimal θ must be positive and strictly less than* 1.

One can provide further characterization of optimal tax and privatization by utilizing the first order conditions (8), (9), (11) and (12) and obtaining the following.

$$t = E' - \frac{p'\left[q_2\frac{\partial q_2^*}{\partial t} + \theta q_1\frac{\partial q_1^*}{\partial t}\right]}{\theta[\frac{\partial a_1^*}{\partial t} - \frac{\partial q_1^*}{\partial t}] + [\frac{\partial a_2^*}{\partial t} - \frac{\partial q_2^*}{\partial t}]}, \tag{27}$$

$$\theta = -\frac{(t - E' - q_2 p')\frac{\partial q_2^*}{\partial \theta}}{(t - E' - q_1 p')\frac{\partial q_1^*}{\partial \theta} + (E' - t)\frac{\partial a_1^*}{\partial \theta}}. \tag{28}$$

These expressions will be useful to consider particular examples and special cases. In general we expect the optimal tax to satisfy the following inequality: $t < E' < t - q_2 p'$. But proving it in the general case seems cumbersome. As this is something peripheral to our interest, we leave it here. An illustrative example (Example 3) is provided in

Appendix B. Two special cases can be derived from our general expressions. Suppose there is no environmental damage, i.e. $E'(.) = 0$. Now in one case we set $t = 0$ exogenously and determine optimal θ, and alternatively set $\theta = 0$ and determine optimal t. Suppose $t = E'(.) = 0$; from (28) we get

$$\theta = -\frac{q_2 \frac{\partial q_2^*}{\partial \theta}}{q_1 \frac{\partial q_1^*}{\partial \theta}}.$$

Since $\frac{\partial q_2^*}{\partial \theta} > 0$ and $\frac{\partial q_1^*}{\partial \theta} < 0$, we must have $\theta > 0$. Further, as a regularity condition we should expect $q_2 < q_1$ and $\frac{\partial (q_1 + q_2^*)}{\partial \theta} < 0$, which implies that $|\frac{\partial q_1^*}{\partial \theta}| > |\frac{\partial q_2^*}{\partial \theta}|$. Hence, $\theta < 1$; that is, privatization must be partial. This is the well-known result of Matsumura (1998). Alternatively, set $\theta = E'(.) = 0$. Then from (27) we get $t = p' q_2 < 0$. That is, the private firm must be subsidized to increase its production to the socially optimal level. This result and its several ramifications have been highlighted in White (1996) and Fjell and Heywood (2004).

5 Foreign firm

Competition between a foreign firm and a state owned firm has been a subject of interest for long time. The foreign firm's profit is a leakage from the national economy and therefore the measure of social welfare changes. In addition the government's objective function diverges from the social welfare when it uses tax and subsidy instruments. These two implications create a paradoxical situation; as we demonstrate below the government does not implement the social optimum, though it is perfectly implementable.

Let firm 2 be owned by a foreign party to the extent of μ, $(0 < \mu < 1)$, causing the μ proportion of firm 2's profit being repatriated to a foreign country. Hence the social welfare would be given by the total benefits from consumption minus the total social cost of production and the leakage to the foreign country. This is given as follows.

$$W^F = W - \mu \left[p q_2 - C_2(q_2) - g_2(a_2) \right] \tag{29}$$

where W is given in (1). Socially optimal outputs and abatements are obtained by maximizing (29) with respect to (q_i, a_i) from the following equations.

$$\frac{\partial W^F}{\partial q_1} = p - C_1'(q_1) - E'(.) - \mu q_2 p'(.) = 0 \tag{30}$$

$$\frac{\partial W^F}{\partial a_1} = -g_1'(a_1) + E'(.) = 0 \tag{31}$$

$$\frac{\partial W^F}{\partial q_2} = (1-\mu)\left[p - C_2'(q_2)\right] - E'(.) - \mu q_2 p'(.) = 0 \tag{32}$$

$$\frac{\partial W^F}{\partial a_2} = -(1-\mu) g_2'(a_2) + E'(.) = 0 \tag{33}$$

Let (q_i^{SF}, a_i^{SF}), $i = 1, 2$, denote the socially optimal output and abatement when the second firm is partly owned in a foreign country. It is noteworthy from Eq. (30) that $p - C_1'(q_1) - E'(.) < 0$. That is, the public firm must produce beyond the point where price is equal to social marginal cost. This parallels the 'below marginal cost' pricing result of the public firm in mixed oligopoly models with foreign firms.[15]

Next, we show that the social optimum as given in (30)-(33) is implementable through a tax-subsidy scheme. Consider the firms' output and abatement choice problem. The partially public firm's objective function is given as before as a weighted average of profit and social welfare. In the present context it becomes

$$O^F = \theta\pi_1 + (1-\theta)W^F = \theta\pi_1 + (1-\theta)W - (1-\theta)\mu[pq_2 - C_2(q_2) - g_2(a_2)].$$

Firm 2's objective function remains unchanged at π_2. Both firm face a tax t on output and a subsidy s on abatement. For simplicity we ignore the abatement constraints, and derive the output and abatement equations as

$$\frac{\partial O^F}{\partial q_1} = [p - C'(q_1) - E'(.)] - \theta[t - E'(.) - q_1 p'(.)] - (1-\theta)\mu q_2 p'(.) = 0 \tag{34}$$

$$\frac{\partial O^F}{\partial a_1} = -g'(a_1) + \theta s + (1-\theta)E'(.) = 0 \tag{35}$$

$$\frac{\partial \pi_2}{\partial q_2} = p + q_2 p'(.) - C_2'(q_2) - t = 0 \tag{36}$$

$$\frac{\partial \pi_2}{\partial a_2} = -g_2'(a_2) + s = 0. \tag{37}$$

[15]In Appendix B we provide an example where we show that when $\mu = 1$ the private firm's output can be zero in the social optimum.

It is now straightforward to see that if the public firm is kept fully public (i.e. $\theta = 0$) and tax and subsidy are so chosen that $s = E'(.)/(1-\mu)$ and $t = [E'(.) + q_2 p'(.)]/(1-\mu)$, then through the firm interactions the socially optimal outputs and abatements are achieved. Substitute these values into (34)-(37), and then these equations become identical to (30)-(33). Let us denote these values of s and t as s^{SF} and t^{SF} respectively.

Proposition 6. *The social optimum is implementable by choosing*

$$\theta = 0, \quad s = s^{SF} \equiv \frac{E'(q_1^{SF} + q_2^{SF} - a_1^{SF} - a_2^{SF})}{(1-\mu)}$$
$$\textit{and} \quad t = t^{SF} \equiv s^{SF} + q_2^{SF} \frac{p'(q_1^{SF} + q_2^{SF})}{(1-\mu)}.$$

Now we examine whether it is optimal for the government to implement the social optimum. In the presence of tax and subsidies the government's objective will differ from the social welfare W^F in the following way:

$$\begin{aligned} G^F &= \left[\int_0^Q p(x)dx - pQ\right] + [pq_1 - C_1(q_1) - g_1(a_1) - tq_1 + sa_1] \\ &\quad + (1-\mu)[pq_2 - C_2(q_2) - g_2(a_2) - tq_2 + sa_2] - E(Q-A) + tQ - sA \\ &= W - \mu[pq_2 - C_2(q_2) - g_2(a_2) - tq_2 + sa_2] \\ &= W^F + \mu[tq_2 - sa_2]. \end{aligned} \tag{38}$$

Thus, it appears that when there is a leakage from the national economy and the government uses tax and/or subsidy, no longer will it remain neutral to these two instruments. It will try to maximize social welfare plus μ proportion of net tax revenues (net of subsidies) collected from the foreign firm. In setting the optimal tax, subsidy and privatization the government takes into account the firms' future output and abatement responses. Noting that a_2 is unaffected by θ or t, and $\frac{\partial a_2}{\partial s} > 0$, $\frac{\partial q_2}{\partial \theta} > 0$, $\frac{\partial q_2}{\partial t} < 0$ and $\frac{\partial q_2}{\partial s} < 0$ we derive the government's choice equations as:

$$\begin{aligned}
\frac{\partial G^F}{\partial \theta} &= \frac{\partial W^F}{\partial \theta} + \mu t \frac{\partial q_2}{\partial \theta} = 0, \\
\frac{\partial G^F}{\partial t} &= \frac{\partial W^F}{\partial t} + \mu q_2 \left[1 + \frac{t}{q_2}\frac{\partial q_2}{\partial t}\right] = 0, \\
\frac{\partial G^F}{\partial s} &= \frac{\partial W^F}{\partial s} + \mu \left[t\frac{\partial q_2}{\partial s} - s\frac{\partial a_2}{\partial s} - a_2\right] = 0.
\end{aligned}$$

To see how the government's choices diverge from the social optimum, let us evaluate these choice equations at the socially optimal (θ, s, t) (at which W^F attains its maximum).

$$\frac{\partial G^F}{\partial \theta}|_{\theta=0} = \mu t \frac{\partial q_2}{\partial \theta} > 0, \tag{39}$$

$$\frac{\partial G^F}{\partial t}|_{t=t^{SF}} = \mu q_2 \left[1 + \frac{t}{q_2}\frac{\partial q_2}{\partial t}\right], \tag{40}$$

$$\frac{\partial G^F}{\partial s}|_{s=s^{SF}} = \mu \left[t\frac{\partial q_2}{\partial s} - s\frac{\partial a_2}{\partial s} - a_2\right] < 0. \tag{41}$$

It is clear that the government's optimal choices do not coincide with socially optimal choices. As the government naturally develops a preference for tax and an aversion for subsidies, it would like to privatize the public firm to some extent, because that will increase the output of firm 2 and thus result in a slightly higher tax revenue. Similarly, it will also choose a smaller subsidy than s^{SF} as is evident from the fact that at s^{SF}, $\frac{\partial G}{\partial s} < 0$. For the tax rate, the government's choice depends on the tax elasticity. If the magnitude of the tax elasticity of the foreign firm's output is less than 1 (i.e. tax-inelastic), the government will set a tax higher than the socially optimal rate t^{SF}.

Proposition 7. *When there is a foreign firm, the government will not implement the socially optimal outputs and abatement, though it is implementable. The government will privatize the public firm (at least partially) and set a smaller subsidy rate than what is socially optimal. It will also set a higher (smaller) tax rate if the output of the foreign firm is tax inelastic (elastic) than what is socially optimal.*

6 Conclusion

This paper examines the optimal tax-subsidy policy in the context of a mixed duopoly with pollution and abatement by considering a wider policy space and fairly general demand and cost functions. It shows that the first best outcome can be achieved by taxing the output at a lower rate than the rate at which pollution abatement is to be subsidized. No-privatization turns out to be socially optimal, and indeed the government finds it optimal to implement the social optimum. However, if the private firm is partially or fully owned by a foreign party, we may have a paradoxical situation. Though the first best is still implementable, the government will not find in its interest to implement the first best. This is because the government develops a preference for tax and aversion to subsidy vis-à-vis the foreign firm. In this case, some privatization will be optimal for the government. These results run contrary to some of the existing papers which show that the social optimum is not generally implementable and privatization is socially optimal.

The paper also provides a generalized treatment of pollution tax with abatement constraints and brings together a number of results from the mixed oligopoly literature and from the environmental economics literature. It is shown in the special case of pollution tax, which restricts the tax on output to be same as the subsidy on abatement, that the government cannot achieve the social optimum, and in the second best the optimal pollution tax needs to be complemented by partial privatization.

Throughout we restricted our analysis to homogeneous products. It might be interesting to extend our work to differentiated products. Further, one may consider alternative environmental policy instruments, such as pollution standards and tradable permits, and examine the relative effectiveness of alternative environmental policy instruments in a mixed oligopoly setup. We leave this for future research.

Appendix

A. Proofs and Derivations

1. Uniqueness of the solution to the Eqs. (4)-(7)**.** We provide a sketch of the proof of the claim that the equilibrium will be unique. More formally, we require the four equations (4)-(7) to have exactly one solution for (q_1, q_2, a_1, a_2). First of all, we note that these equations are continuous. Next, differentiate the equations further by (q_1, q_2, a_1, a_2) and consider the the Jacobian matrix of the resulting equations. This Jacobian matrix, we assume, is non-singular, so that by the Implicit Function Theorem we can argue that (q_1, q_2, a_1, a_2) are all functions of (s, t) and other parameters of the model. This establishes local uniqueness.

However, for global uniqueness and for a more direct approach, we take the following line of argument. Let us consider Eqs (4)-(5) and Eqs. (6)-(7) separately, as they arise from two separate optimization. First, we check from (6)-(7) that a_2 is independent of q_1 any other strategy, while q_2 is a strategic substitute for q_1, but unrelated to a_1. By using the standard techniques of comparative statics we derive from (6)-(7)

$$\frac{\partial q_2}{\partial q_1} = -\frac{p'(.) + q_2 p''(.)}{\partial^2 \pi_2 / \partial q_2^2} < 0 \qquad \frac{\partial q_2}{\partial a_1} = \frac{\partial a_2}{\partial q_1} = \frac{\partial a_2}{\partial a_1} = 0.$$

Now consider Eqs. (4)-(5). Carrying out the comparative statics in terms of q_2 and a_2 we get

$$\begin{aligned}
\frac{\partial q_1}{\partial q_2} &= \underbrace{\frac{1}{D}}_{>0}\left[\left\{\underbrace{p'(.) + \theta q_1 p''(.)}_{<0}\right\}\left\{\underbrace{g_1''(a_1) + (1-\theta)E''(.)}_{>0}\right\} - \underbrace{(1-\theta)E''(.)g_1''(a_1)}_{>0}\right] < 0 \\
\frac{\partial q_1}{\partial a_2} &= \frac{1}{D}(1-\theta)g_1''(a_1)E''(.) > 0 \\
\frac{\partial a_1}{\partial q_2} &= \frac{1}{D}(1-\theta)E''(.)\left[C_1''(q_1) - \theta p'(.)\right] > 0 \\
\frac{\partial a_1}{\partial a_2} &= \frac{1}{D}(1-\theta)E''(.)\left[\underbrace{p'(.)(1+\theta) - C_1''(q_1) + \theta q_1 p''(.)}_{<0}\right] < 0
\end{aligned}$$

where

$$
\begin{aligned}
D &= \frac{\partial^2 O}{\partial q_1^2}\frac{\partial^2 O}{\partial a_1^2} - \left(\frac{\partial^2 O}{\partial q_1 \partial a_1}\right)^2 \\
&= \Big[\underbrace{-p'(.) + C_1''(q_1) - \theta p'(.) - \theta q_1 p''(.)}_{>0}\Big][g_1''(.) + (1-\theta)E''(.)] + (1-\theta)g_1''(.)E''(.) > 0.
\end{aligned}
$$

As can be seen from the above a_1 is strategic substitute to a_2 for any given (q_1, q_2). On the (a_1, a_2) plane we can draw the reaction curve of the firm 2, which will be a flat line,say at a_2^*, and the reaction curve of firm 1, which will be downward sloping. With standard boundary restrictions on the reaction function, we can ensure that there is a unique (a_1^*, a_2^*) for any given (q_1, q_2). Then we also see that for any given (a_1, a_2), q_1 and q_2 are strategic substitutes to each other. Therefore, we will have standard downward sloping output reaction curves on the (q_1, q_2) plane for any (a_1, a_2). Hence with standard boundary and slope conditions we will get unique intersection of the output reaction curves. When both pairs of the reaction curves correspond to the equilibrium abatements and outputs, we have the overall equilibrium. It is unique because each pair of reaction curves cross only once.

We should also note that if $\theta = 1$ then both a_1 and a_2 are independent of any other actions. Then given these unique (a_1, a_2) we determine the uniqueness of output equilibrium by appealing to the standard unique equilibrium of the Cournot duopoly.

2. Proof of Proposition 1. Suppose $C_1(.) = C_2(.) = C(.)$ and $g_1(.) = g_2(.) = g(.)$. By assumption $C''(.) \geq 0$, $g''(.) \geq 0$. Suppose strict inequality holds for at least one. Contrary to our claim, assume that the socially optimal tax rate induces $q_1 > 0, q_2 = a_2 = 0$. Let the socially optimal output be denoted as q_1^*. Social welfare is $W^* = \int_0^{q_1^*} p(x)dx - C(q_1^*) - g(a_1^*) - E(q_1^* - a_1^*)$. Now redistribute q_1^* as $q_1 = q_1^* - \delta$ $(0 < \delta < q_1^*)$ and $q_2 = \delta$, and a_1^* as $a_1 = a_1^* - \epsilon$ $(0 < \epsilon < a_1)$ and $a_2 = \epsilon$. Then $W(\delta, \epsilon) = \int_0^{q_1^*} p(x)dx - C(q_1^* - \delta) - C(\delta) - g(a_1^* - \epsilon) - g(\epsilon) - E(q_1^* - a_1^*)$. Since by assumption $C''(.) \geq 0, g''(.) \geq 0$ with at least one strict inequality, we must have $C(q_1^*) \geq C(q_1^* - \delta) + C(\delta), g(a_1^*) \geq g(a_1^* - \epsilon) + g(\epsilon)$ with at least one strict inequality. Therefore, $W(\delta,\epsilon)) > W^*$, which is a contradiction to the premise that W^* was the maximal value of W. Therefore, q_2 must be positive.

By corollary, if $C''(q_i) = 0$, $g''(a_i) = 0$ for $i = 1, 2$, then for any (δ,ϵ), we have $C(q_1^*) = C(q_1^* - \delta) + C(\delta), g(a_1^*) = g(a_1^* - \epsilon) + g(\epsilon)$, and therefore $W(\delta,\epsilon) = W^*$. Hence, $q_2 = 0$ will be optimal. **Q.E.D.**

3. Proof of Proposition 2. Set $\theta = 0$ and substitute $t = E'(q_1^S + q_2^S - a_1^2 - a_2^S) + q_2^S p'(q_1^S + q_2^S)$ in (6) and $s = E'(q_1^S + q_2^S - a_1^2 - a_2^S)$ in (7). Eqs. (4) and (6) coincide with (2) $(i = 1, 2)$ and Eqs. (5) and (7) coincide with (3) $(i = 1, 2)$. Hence, the suggested tax and subsidy will induce socially optimal outputs and abatements. The government will also find it optimal to choose this tax and subsidy because the government's objective function is same as W. **Q.E.D**

2. Derivation of (16)**.** From (12) we get $\frac{\partial a_2^*}{\partial t} = \frac{1}{g_2''(a_2)} >$. Now differentiate (8), (9) and (11) to obtain the following expressions:

$$\frac{\partial^2 W}{\partial q_1^2}\frac{\partial q_1^*}{\partial t} + \frac{\partial^2 W}{\partial a_1 \partial q_1}\frac{\partial a_1^*}{\partial t} + \frac{\partial^2 W}{\partial q_2 \partial q_1}\frac{\partial q_2^*}{\partial t} + \frac{\partial^2 W}{\partial a_2 \partial q_1}\frac{\partial a_2^*}{\partial t} = 0$$
$$\frac{\partial^2 W}{\partial q_1 \partial a_1}\frac{\partial q_1^*}{\partial t} + \frac{\partial^2 W}{\partial a_1^2}\frac{\partial a_1^*}{\partial t} + \frac{\partial^2 W}{\partial q_2 \partial a_1}\frac{\partial q_2^*}{\partial t} + \frac{\partial^2 W}{\partial a_2 \partial a_1}\frac{\partial a_2^*}{\partial t} = 0$$
$$\frac{\partial^2 \pi_2}{\partial q_1 \partial q_2}\frac{\partial q_1^*}{\partial t} + \frac{\partial^2 \pi_2}{\partial a_1 \partial q_2}\frac{\partial a_1^*}{\partial t} + \frac{\partial^2 \pi_2}{\partial q_2^2}\frac{\partial q_2^*}{\partial t} + \frac{\partial^2 \pi_2}{\partial a_2 \partial q_2}\frac{\partial a_2^*}{\partial t} = 1.$$

From Eqs. (8), (9) and (11) the following can be derived:

$$\begin{aligned}
\frac{\partial^2 W}{\partial q_2 \partial q_1} &= p'(.) - E''(.) < 0, \quad \frac{\partial^2 W}{\partial a_1^2} = -[g_1''(a_1) + E''(.)] < 0,\\
\frac{\partial^2 W}{\partial q_1^2} &= p'(.) - C_1''(q_1) - E''(.) < 0, \quad \frac{\partial^2 \pi_2}{\partial q_2^2} = 2p'(.) + q_2 p''(.) - C_2''(q_2) < 0\\
\frac{\partial^2 \pi_2}{\partial q_1 q_2} &= p'(.) + q_2 p''(.) < 0, \quad \frac{\partial^2 \pi_2}{\partial a_1 \partial q_2} = \frac{\partial^2 \pi_2}{\partial a_2 \partial q_2} = 0,
\end{aligned}$$

and

$$\frac{\partial^2 W}{\partial a_1 \partial q_1} = \frac{\partial^2 W}{\partial a_2 \partial q_1} = \frac{\partial^2 W}{\partial q_2 \partial a_1} = \frac{\partial^2 W}{\partial q_1 \partial a_1} = -\frac{\partial^2 W}{\partial a_2 \partial a_1} = E''(.) > 0.$$

Substituting these relations and the expression for $\frac{\partial a_2^*}{\partial t}$ into the above system of equations, we rewrite it as

$$\begin{aligned}
&\frac{\partial^2 W}{\partial q_1^2}\frac{\partial q_1^*}{\partial t} + E''(.)\frac{\partial a_1^*}{\partial t} + \Big[p'(.) - E''(.)\Big]\frac{\partial q_2^*}{\partial t} = -\frac{E''(.)}{g_2''(a_2)}\\
&E''(.)\frac{\partial q_1^*}{\partial t} - \Big[g_1'(a_1) + E''(.)\Big]\frac{\partial a_1^*}{\partial t} + E''(.)\frac{\partial q_2^*}{\partial t} = \frac{E''(.)}{g_2''(a_2)}\\
&\frac{\partial^2 \pi_2}{\partial q_1 \partial q_2}\frac{\partial q_1^*}{\partial t} + \frac{\partial^2 \pi_2}{\partial q_2^2}\frac{\partial q_2^*}{\partial t} = 1.
\end{aligned}$$

Stability of Nash equilibrium requires

$$\Delta_1 = \tfrac{\partial^2 W}{\partial q_1^2} < 0, \qquad \Delta_2 = \begin{vmatrix} \frac{\partial^2 W}{\partial q_1^2} & E''(.) \\ E''(.) & \frac{\partial^2 W}{\partial a_1^2} \end{vmatrix} > 0,$$

and

$$\Delta = \begin{vmatrix} \frac{\partial^2 W}{\partial q_1^2} & E''(.) & [p'(.) - E''(.)] \\ E''(.) & \frac{\partial^2 W}{\partial a_1^2} & E''(.) \\ \frac{\partial^2 \pi_2}{\partial q_1 \partial q_2} & 0 & \frac{\partial^2 \pi_2}{\partial q_2^2} \end{vmatrix} < 0.$$

Further we assume that $|\frac{\partial^2 \pi_2}{\partial q_2^2}| > |\frac{\partial^2 \pi_2}{\partial q_2 \partial q_1}|$.

$$\begin{aligned}
\frac{\partial q_1^*}{\partial t} &= \frac{1}{\Delta}\left[E''(.)\frac{g_1'(a_1)}{g_2'(a_2)}\frac{\partial^2 \pi_2}{\partial q_2^2} + p'(.)\{g_1''(a_1) + E''(.)\} - E''(.)g_2''(a_2)\right] > 0\\
\frac{\partial a_1^*}{\partial t} &= \frac{1}{\Delta}\left[\frac{E''(.)}{g_2''(a_2)}\left\{p'(.)\left(\frac{\partial^2 \pi_2}{\partial q_2^2} - \frac{\partial^2 \pi_2}{\partial q_1 \partial q_2}\right) - C_1''(q_1)\frac{\partial^2 \pi_2}{\partial q_2^2}\right\} + E''(.)C_1''(q_1)\right] < 0\\
\frac{\partial q_2^*}{\partial t} &= -\frac{1}{\Delta}\left[\frac{\partial^2 W}{\partial q_1^2}g_1''(a_1) + E''(.)\{p'(.) - C_1''(q_1)\} + \frac{E''}{g_2''(a_2)}\frac{\partial^2 \pi_2}{\partial q_1 \partial q_2}g_1''(a_1)\right] < 0.
\end{aligned}$$

5. Proof of Lemma 4. (a) First consider the effects of an increase in t on (q_1, a_1, q_2). For this we need to totally differentiate (8), (9) and (11) simultaneously and obtain the following expressions:

$$\begin{aligned}
&\frac{\partial^2 O}{\partial q_1^2}\frac{\partial q_1^*}{\partial t} + \frac{\partial^2 O}{\partial a_1 \partial q_1}\frac{\partial a_1^*}{\partial t} + \frac{\partial^2 O}{\partial q_2 \partial q_1}\frac{\partial q_2^*}{\partial t} + \frac{\partial^2 O}{\partial a_2 \partial q_1}\frac{\partial a_2^*}{\partial t} = \theta\\
&\frac{\partial^2 O}{\partial q_1 \partial a_1}\frac{\partial q_1^*}{\partial t} + \frac{\partial^2 O}{\partial a_1^2}\frac{\partial a_1^*}{\partial t} + \frac{\partial^2 O}{\partial q_2 \partial a_1}\frac{\partial q_2^*}{\partial t} + \frac{\partial^2 O}{\partial a_2 \partial a_1}\frac{\partial a_2^*}{\partial t} = -\theta\\
&\frac{\partial^2 \pi_2}{\partial q_1 \partial q_2}\frac{\partial q_1^*}{\partial t} + \frac{\partial^2 \pi_2}{\partial a_1 \partial q_2}\frac{\partial a_1^*}{\partial t} + \frac{\partial^2 \pi_2}{\partial q_2^2}\frac{\partial q_2^*}{\partial t} + \frac{\partial^2 \pi_2}{\partial a_2 \partial q_2}\frac{\partial a_2^*}{\partial t} = 1.
\end{aligned}$$

Further the second order conditions of the equation system constituted by (8), (9), (11) and (12) the following can be derived:

$$
\begin{aligned}
\frac{\partial^2 O}{\partial q_1^2} &= p' - C_1''(q_1) - (1-\theta)E'' + \theta[q_1 p'' + p'] < 0 \\
\frac{\partial^2 O}{\partial q_2 \partial q_1} &= p' - (1-\theta)E'' + \theta q_1 p'' < 0 \\
\frac{\partial^2 O}{\partial a_1 \partial q_1} &= \frac{\partial^2 O}{\partial a_2 \partial q_1} = \frac{\partial^2 O}{\partial q_1 \partial a_1} = \frac{\partial^2 O}{\partial q_2 \partial a_1} = -\frac{\partial^2 O}{\partial a_2 \partial a_1} = (1-\theta)E'' > 0 \\
\frac{\partial^2 O}{\partial a_1^2} &= -g_1''(a_1) - (1-\theta)E'' < 0 \\
\frac{\partial^2 \pi_2}{\partial q_2^2} &= 2p' + q_2 p'' - C_2''(q_2) < 0 \\
\frac{\partial^2 \pi_2}{\partial q_1 \partial q_2} &= p' + q_2 p'' < 0 \\
\frac{\partial^2 \pi_2}{\partial a_2^2} &= -g_2''(a_2) < 0 \\
\frac{\partial^2 \pi_2}{\partial a_1 \partial q_2} &= \frac{\partial^2 \pi_2}{\partial a_2 \partial q_2} = \frac{\partial^2 \pi_2}{\partial q_1 \partial a_2} = \frac{\partial^2 \pi_2}{\partial q_2 \partial a_2} = \frac{\partial^2 \pi_2}{\partial a_1 \partial a_2} = 0
\end{aligned}
$$

Substituting these relations and the expression for $\frac{\partial a_2^*}{\partial t}$ as obtained earlier into the above system of equations, we rewrite it as

$$
\frac{\partial^2 O}{\partial q_1^2}\frac{\partial q_1^*}{\partial t} + (1-\theta)E''\frac{\partial a_1^*}{\partial t} + \Big[p' - (1-\theta)E'' + \theta q_1 p''\Big]\frac{\partial q_2^*}{\partial t} = \theta - \frac{(1-\theta)E''}{g_2''(a_2)} \quad (42)
$$

$$
(1-\theta)E''\frac{\partial q_1^*}{\partial t} - \Big[g_1'(a_1) + (1-\theta)E''\Big]\frac{\partial a_1^*}{\partial t} + (1-\theta)E''\frac{\partial q_2^*}{\partial t} = -\theta + \frac{(1-\theta)E''}{g_2''(a_2)} \quad (43)
$$

$$
\Big[p' + q_2 p''\Big]\frac{\partial q_1^*}{\partial t} + 0 + \Big[2p' + q_2 p'' - C_2''(q_2)\Big]\frac{\partial q_2^*}{\partial t} + 0 = 1. \quad (44)
$$

Stability of Nash equilibrium requires

$$
\Delta_1 = \tfrac{\partial^2 O}{\partial q_1^2} < 0, \qquad \Delta_2 = \begin{vmatrix} \frac{\partial^2 O}{\partial q_1^2} & (1-\theta)E'' \\ (1-\theta)E'' & \frac{\partial^2 O}{\partial a_1^2} \end{vmatrix} > 0,
$$

$$
\text{and } \Delta = \begin{vmatrix} \theta - \frac{(1-\theta)E''}{g_2''(a_2)} & (1-\theta)E'' & [p' - (1-\theta)E'' + \theta q_1 p''] \\ -\theta + \frac{(1-\theta)E''}{g_2''(a_2)} & -[g_1'(a_1) + (1-\theta)E''] & (1-\theta)E'' \\ 1 & 0 & 2p' + q_2 p'' - C_2''(q_2) \end{vmatrix} < 0.
$$

From Eqs. (42)-(44), using the above relations and relations derived from Eqs. (8),(9) and (11), we find the following.

$\frac{\partial q_1^*}{\partial t} = \frac{N_1}{\Delta}$, where $\Delta < 0$ and $N_1 = g_1''(a_1)[p' + \theta q_1 p'' - (1-\theta)E''] + (1-\theta)(p' + \theta q_1 p'')E'' - [2p' + q_2 p'' - C_2''(q_2)]g_1''(a_1)[\theta - \frac{(1-\theta)E''}{g_2''(a_2)}]$.

Now, $N_1 = 0$, if $\theta = \theta_- = \frac{-y-\sqrt{y^2-4xz}}{2x}$ or $\theta = \theta_1 = \frac{-y+\sqrt{y^2-4xz}}{2x}$, where
$x = -q_1 p'' E'' > 0$,
$y = g_1''(a_1)E'' - p'E'' - g_1''(a_1)[1 + \frac{E''}{g_2''(a_2)}][2p' + q_2 p'' - C_2''(q_2)] > 0$, and
$z = g_1''(a_1)\frac{E''}{g_2''(a_2)}[2p' + q_2 p'' - C_2''(q_2)] + g_1''(a_1)(p' - E'') + p'E'' < 0$. Clearly, $\theta_- < 0$ and $0 < \theta_1 < 1$.

It is easy to check that (a) $N_1 < 0$, if $\theta = 0$ and (b) $N_1 > 0$, if $\theta = 1$.

Now, $\frac{\partial^2 N_1}{\partial \theta^2} = -2q_1 p'' E'' > 0$. It implies that N_1 has a minimum.

Therefore, we have (a) $N_1 < 0$ if $0 \leq \theta < \theta_1$, (b) $N_1 = 0$ if $\theta = \theta_1$ and (c) $N_1 > 0$ if $\theta_1 < \theta \leq 1$. It implies (22), since $\Delta < 0$.

We have, $\frac{\partial q_2^*}{\partial t} = \frac{N_2}{\Delta}$, where $\Delta < 0$ and $N_2 = (p' + q_2 p'')g_1''(a_1)[\theta - \frac{(1-\theta)E''}{g_2''(a_2)}] - [g_1''(a_1) + (1-\theta)E''][p' - C_1''(q_1) - (1-\theta)E'' + \theta q_1 p'' + \theta p'] - [(1-\theta)E'']^2$.

If $\theta = 1$, $N_2 = -g_1''(a_1)[p' + (q_1 - q_2)p'' - C_1''(q_1)] > 0$. Because, $q_1^*(t, \theta = 1) = q_2^*(t, \theta = 1)$.

If $\theta = 0$, $N_2 = -\frac{g_1''(a_1)}{g_2''(a_2)}E''[p' + q_2 p''] - g_1''(a_1)[p' - C''(q_1)] - E''[p' - C_1''(q_1)] + E''g_1''(a_1) > 0$.

Moreover, $\frac{\partial^2 N_2}{\partial \theta^2} = 2E''[q_1 p'' + p'] < 0$. It implies that N_2 has a maximum.

Therefore, we have $\frac{\partial q_2^*}{\partial t} < 0 \ \forall \theta \in [0, 1]$.

Finally, $\frac{\partial a_1^*}{\partial t} = \frac{N_3}{\Delta}$, where $\Delta < 0$ and $N_3 = \begin{vmatrix} \frac{\partial^2 O}{\partial q_1^2} & \theta - \frac{(1-\theta)E''}{g_2''(a_2)} & [p' - (1-\theta)E'' + \theta q_1 p''] \\ (1-\theta)E'' & -[\theta - \frac{(1-\theta)E''}{g_2''(a_2)}] & (1-\theta)E'' \\ p' + q_2 p'' & 1 & 2p' + q_2 p'' - C_2''(q_2) \end{vmatrix}$.

If $\theta = 0$, $N_3 = (p' - E'')\frac{E''}{g_2''(a_2)}[p' - C_2''(q_2)] + C_1''(q_1)E'' - E''\frac{E''}{g_2''(a_2)}(p' + q_2 p'') > 0$.

If $\theta = 1$, $N_3 = [3p' + 2q_1 p'' - C_1''(q_1)][-p' + C_1''(q_1)] < 0$

$\frac{\partial^2 N_2}{\partial \theta^2} = 2(p' + q_1 p'')[1 + \frac{E''}{g_2''(a_2)}][-p' + C_2''(q_2)] < 0$, which implies that N_3 has a maximum.

Since $N_3(\theta = 1) < 0 < N_3(\theta = 0)$ and $N_3(\theta)$ has a maximum, $\ni \theta = \theta_2 \in (0, 1)$ such that $N_3(\theta_2) = 0$.

Therefore, we have (a) $N_3 > 0$ if $0 \le \theta < \theta_2$, (b) $N_3 = 0$ if $\theta = \theta_2$ and (c) $N_1 < 0$ if $\theta_2 < \theta \le 1$. It implies (23), since $\Delta < 0$. This completes part (a) of the proposition.

(b) Now consider part (b) of the proposition where we examine the impacts of θ. Differentiating (8), (9), (11) and (12) with respect to θ we get the following.

$$\frac{\partial^2 O}{\partial q_1^2}\frac{\partial q_1^*}{\partial \theta} + \frac{\partial^2 O}{\partial a_1 \partial q_1}\frac{\partial a_1^*}{\partial \theta} + \frac{\partial^2 O}{\partial q_2 \partial q_1}\frac{\partial q_2^*}{\partial \theta} + \frac{\partial^2 O}{\partial a_2 \partial q_1}\frac{\partial a_2^*}{\partial \theta} = -[q_1 p' - t + E']$$
$$\frac{\partial^2 O}{\partial q_1 \partial a_1}\frac{\partial q_1^*}{\partial \theta} + \frac{\partial^2 O}{\partial a_1^2}\frac{\partial a_1^*}{\partial \theta} + \frac{\partial^2 O}{\partial q_2 \partial a_1}\frac{\partial q_2^*}{\partial \theta} + \frac{\partial^2 O}{\partial a_2 \partial a_1}\frac{\partial a_2^*}{\partial \theta} = -[t - E']$$
$$\frac{\partial^2 \pi_2}{\partial q_1 \partial q_2}\frac{\partial q_1^*}{\partial \theta} + \frac{\partial^2 \pi_2}{\partial a_1 \partial q_2}\frac{\partial a_1^*}{\partial \theta} + \frac{\partial^2 \pi_2}{\partial q_2^2}\frac{\partial q_2^*}{\partial \theta} + \frac{\partial^2 \pi_2}{\partial a_2 \partial q_2}\frac{\partial a_2^*}{\partial \theta} = 0$$
$$\frac{\partial^2 \pi_2}{\partial q_1 \partial a_2}\frac{\partial q_1^*}{\partial \theta} + \frac{\partial^2 \pi_2}{\partial a_1 \partial a_2}\frac{\partial a_1^*}{\partial \theta} + \frac{\partial^2 \pi_2}{\partial q_2 \partial a_2}\frac{\partial q_2^*}{\partial \theta} + \frac{\partial^2 \pi_2}{\partial a_2^2}\frac{\partial a_2^*}{\partial \theta} = 0$$

We already know that $\frac{\partial a_2^*}{\partial \theta} = 0$, since $\frac{\partial^2 \pi_2}{\partial q_1 \partial a_2} = \frac{\partial^2 \pi_2}{\partial q_2 \partial a_2} = \frac{\partial^2 \pi_2}{\partial a_1 \partial a_2} = 0$ and $\frac{\partial^2 \pi_2}{\partial a_2^2} < 0$. Therefore, we have the following system of equations.

$$\frac{\partial^2 O}{\partial q_1^2}\frac{\partial q_1^*}{\partial \theta} + \frac{\partial^2 O}{\partial a_1 \partial q_1}\frac{\partial a_1^*}{\partial \theta} + \frac{\partial^2 O}{\partial q_2 \partial q_1}\frac{\partial q_2^*}{\partial \theta} = -[q_1 p' - t + E']$$
$$\frac{\partial^2 O}{\partial q_1 \partial a_1}\frac{\partial q_1^*}{\partial \theta} + \frac{\partial^2 O}{\partial a_1^2}\frac{\partial a_1^*}{\partial \theta} + \frac{\partial^2 O}{\partial q_2 \partial a_1}\frac{\partial q_2^*}{\partial \theta} = -[t - E']$$
$$\frac{\partial^2 \pi_2}{\partial q_1 \partial q_2}\frac{\partial q_1^*}{\partial \theta} + 0 + \frac{\partial^2 \pi_2}{\partial q_2^2}\frac{\partial q_2^*}{\partial \theta} = 0$$

Solving the above system of equations we get,

$\frac{\partial q_1^*}{\partial \theta} = \frac{1}{\Delta}\frac{\partial^2 \pi_2}{\partial q_2^2}[(1-\theta)E'' q_1 p' - g_1''(t - E' - q_1 p')] < 0$

$\frac{\partial q_2^*}{\partial \theta} = \frac{1}{\Delta}\frac{\partial^2 \pi_2}{\partial q_1 \partial q_2}[-(1-\theta)(t - E')E' + (t - q_1 p' - E')\{g_1''(a_1) + (1-\theta)E''\}] > 0$

$\frac{\partial a_1^*}{\partial \theta} = \frac{1}{\Delta}[\{-p' + C_2''(q_2)\}\{(t - E' - q_1 p')(1-\theta)E'' - (E' - t)(p' + \theta q_1 p'' - (1-\theta)E'')\} + \{2p' + q_2 p'' - C_2''(q_2)\}(E' - t)\{\theta p' - C_1''(q_1)\}] < 0$,

since $\Delta < 0$ and $t - E' - q_1 p' > 0$. **Q.E.D.**

6. Proof of Proposition 5. To see that $t^* \le E'(.)$ consider the firms' choice equations (8), (9),(11) and (12), where we set $\lambda_1 = \lambda_2 = 0$. Note that if $t > E'(.)$ the resultant output and abatements will be $q_1 \le q_1^S$, $q_2 < q_2^S$, $a_1 \ge a_1^S$ and $a_2 > a_2^S$, at all $\theta \ge 0$. This cannot be optimal, because by reducing t to $E'(.)$ the government can always induce $a_1 = a_1^S$, $a_2 = a_2^S$, and at the same time increase increase at least q_2 (and q_1 as well if $\theta > 0$). This will clearly increase the social welfare; hence $t^* > E'(.)$ cannot be optimal. Therefore, we must have $t \le E'(.)$.

Now consider equation (26) for optimal θ. We know at all θ, $\frac{\partial a_2^*}{\partial \theta} = 0$, $\frac{\partial W}{\partial q_2} = p - C_2'(q_2) - E'(.) > 0$ (for $q_2 \le q_1$ and $t \le E'(.)$) and $\frac{\partial q_2^*}{\partial \theta} > 0$. Now evaluate the expression at $\theta = 0$. Since at $\theta = 0$

$$\frac{\partial W}{\partial q_1} = 0, \quad \frac{\partial W}{\partial a_1} = 0, \quad \text{we have}$$

$$\frac{\partial W(.)}{\partial \theta}|_{\theta=0} = [p - C_2'(q_2) - E'(.)] \frac{\partial q_2^*}{\partial \theta} > 0.$$

Therefore, $\theta = 0$ cannot be optimal. Next evaluate the same at $\theta = 1$. Due to symmetry we will now have $q_1 = q_2$ and thus,

$$\begin{aligned} \frac{\partial W(.)}{\partial \theta}|_{\theta=1} &= \frac{\partial W}{\partial q_1}\frac{\partial q_1^*}{\partial \theta} + \frac{\partial W}{\partial a_1}\frac{\partial a_1^*}{\partial \theta} + \frac{\partial W}{\partial q_2}\frac{\partial q_2^*}{\partial \theta} \\ &= [p - C_1'(q_1) - E'(.)]\left[\frac{\partial q_1^*}{\partial \theta} + \frac{\partial q_2^*}{\partial \theta}\right] + [-g_1'(a_1) + E'(.)]\frac{\partial a_1}{\partial \theta}. \end{aligned}$$

By the regularity condition, industry output must expand with a fall in privatization; therefore, $\frac{\partial(q_1+q_2)}{\partial \theta} = \frac{\partial q_1^*}{\partial \theta} + \frac{\partial q_2^*}{\partial \theta} < 0$. Also, $-g_1'(a_1) + E'(.) \ge 0$ because at $\theta = 1$ the fully privatized firm will choose a_1 by the equation $-g_1'(a_1) + t = 0$, and we have already established that $t \le E'(.)$. Hence, $-g_1'(a_1) + E'(.) \ge 0$. Finally, it has also been noted that $\frac{\partial a_1^*}{\partial \theta} < 0$. Hence,

$$\frac{\partial W(.)}{\partial \theta}|_{\theta=1} < 0.$$

Hence, $\theta = 1$ cannot be optimal either. Therefore, optimal θ must lie within the open interval $(0, 1)$. **Q.E.D.**

B. Examples

1. Pollution tax and full public ownership

We consider two examples. In both examples we assume linear demand curves, strictly convex abatement cost and aggregate emission functions. However, in the first example we consider constant marginal cost of production. In the second example we allow the marginal production cost to be increasing as well. However, for the second example due to the difficulty of obtaining any tractable solution we consider a numerical example.

Example 1: Suppose $p = B - b(q_1 + q_2)$, $C(q_i) = cq_i$, $g(a_i) = \frac{\gamma}{2}a_i^2$ and $E(.) = \frac{e}{2}(q_1 + q_2 - a_1 - a_2)^2$. That is, $C''(.) = 0$, but $g''(.) > 0$.

The Nash equilibrium outputs and abatements subject to the constraints $q_1 \geq a_1$ and $q_2 \geq a_2$ are obtained by maximizing W and π_2. Further, assume that the constraint of firm 1 will not bind and the constraint of firm 2 may bind. Hence, the equilibrium outputs and abatements are given by:

$$\begin{aligned}
q_1 &: \quad B - c + e\ (a_1 + a_2) - (b + e)\ (q_1 + q_2) = 0 \\
a_1 &: \quad -(\gamma\, a_1) - e\ (a_1 + a_2) + e\ (q_1 + q_2) = 0 \\
q_2 &: \quad B - c - t - b\, q_1 - 2b\, q_2 + \lambda_2 = 0 \\
a_2 &: \quad t - \gamma\, a_2 - \lambda_2 = 0 \\
&\quad \lambda_2[q_2 - a_2] = 0.
\end{aligned}$$

First assume that the constraint does not bind (i.e. $\lambda_2 = 0$). Denote $q_1^* + q_2^* = Q^*$, $a_1^* + a_2^* = A^*$, $t/\gamma = t^0$ and $B - c = B^0$ and obtain the Nash equilibrium abatements and outputs as follows.

$$\begin{aligned}
a_1^* &= \frac{e\ (Q^* - t^0)}{e + \gamma}, \quad a_2^* = t^0, \\
q_1^* &= \frac{B^0\{b\ (e + \gamma) - e\,\gamma\} + t\,\{3\,b\,e + (b + e)\ \gamma\}}{b\,\{e\,\gamma + b\ (e + \gamma)\}}, \\
q_2^* &= \frac{B^0\,e\,\gamma - t\,\{2\,b\,e + (b + e)\ \gamma\}}{b\,\{e\,\gamma + b\ (e + \gamma)\}}, \\
Q^* &= \frac{B^0 b\ (e + \gamma) + t\,b\,e}{b\,\{e\,\gamma + b\ (e + \gamma)\}}.
\end{aligned}$$

It is straightforward to see that with an increase in t a_1^* falls and q_1^* rises, while a_2^* rises and q_2^* falls as claimed in Lemma 1. Further, the aggregate abatement will also rise and total social damage will fall with t. Now substitute the above abatement and output functions in the social welfare function and derive the optimal tax rate as

$$t = \frac{B^0\, e\, \gamma}{2\, b\, e + (b+e)\ \gamma}.$$

But at this tax rate $q_2^* = 0$. But we know that this cannot be optimal (Proposition 1), as it violates the abatement constraint for firm 2. To satisfy the abatement constraint, t must be reduced from the value of t given above.

So the abatement constraint of firm 2 must bind; in fact it would just bind ($a_2^* = q_2^*$). When that is taken into account, the socially optimal t would be simply $t^* = \gamma q_2^*$. Substituting the expression of q_2^* we get

$$t^* = \frac{B^0 e\gamma^2}{b\{e\gamma + b(e+\gamma)\} + \gamma\{2be + (b+e)\gamma\}}.$$

Example 2: In this example we let the marginal cost of production to be increasing. But we consider a numerical example for ease of exposition. Suppose $p = 10 - (q_1 + q_2)$, $C(q_i) = q_i + \frac{k}{2} q_i^2$, $g(a_i) = \frac{1}{10} a_i^2$ and $E(.) = \frac{1}{2}(q_1 + q_2 - a_1 - a_2)^2$.

Consider the second stage problem. Given any tax rate t, the equilibrium outputs and abatements are given by:

$$q_1:\ 9 + a_1 + a_2 - (2+k)\ q_1 - 2\, q_2 \lambda_1 = 0 \tag{E2.1}$$

$$a_1:\ -6a_1 - 5a_2 + 5(q_1 + q_2) - \lambda_1 = 0, \tag{E2.2a}$$

$$\lambda_1 [q_1 - a_1] = 0, \tag{E2.2b}$$

$$q_2:\ 9 - t - q_1 - 2\, q_2 - k\, q_2 + \lambda_2 = 0 \tag{E2.3}$$

$$a_2:\ 5\, t - a_2 - \lambda_2 = 0, \tag{E2.4a}$$

$$\lambda_2 [q_2 - a_2] = 0. \tag{E2.4b}$$

Assuming $a_1 < q_1$ and $a_2 < q_2$ (and thereby setting $\lambda_1 = \lambda_2 = 0$), from (E2.1), (E2.2a), (E2.3) and (E2.4a) we get,

$$\begin{aligned} q_1^* &= \frac{45 + 17\,t + k\ (54 + 5\,t)}{7 + k\ (19 + 6\,k)} \\ q_2^* &= \frac{9 + 54\,k - 6\ (2 + k)\ t}{7 + k\ (19 + 6\,k)} \\ a_1^* &= \frac{45 + 90\,k - 5\ (5 + k\ (16 + 5\,k))\ t}{7 + k\ (19 + 6\,k)} \\ a_2^* &= 5\,t \end{aligned} \tag{E2.5}$$

From the above expressions, it is clear that $\frac{\partial q_1}{\partial t} > 0$, $\frac{\partial a_1}{\partial t} < 0$ and $\frac{\partial q_2}{\partial t} < 0$, $\frac{\partial a_2}{\partial t} > 0$. Also, we can identify two critical values of t at which the two abatement constraints just bind (separately).

$$\begin{aligned} a_1^* = q_1^* \quad &\text{when} \quad t = \frac{36\,k}{42 + 85\,k + 25\,k^2}\ (= t_1,\ \text{say}) \\ a_2^* = q_2^* \quad &\text{when} \quad t = \frac{9 + 54\,k}{47 + k\ (101 + 30\,k)}\ (= t_2,\ \text{say});\ t_2 > t_1. \end{aligned}$$

Further, when $a_1 = q_1$ and $a_2 < q_2$, (q_1, q_2) is solved from the following two equations.

$$\begin{aligned} 9 + 5t - (1 + k)q_1 - 2q_2 &= 0 \\ 9 - t - q_1 - (2 + k)q_2 &= 0. \end{aligned}$$

The solution to these equations are:

$$\begin{aligned} \tilde{q}_1 &= \frac{9(3 - k - k^2) + t(9 + 7k + k^2)}{3 + k}, \\ \tilde{q}_2 &= \frac{9k - (6 + k)t}{3 + k} \\ \tilde{a}_2 &= 5t, \quad \tilde{a}_1 = \tilde{q}_1. \end{aligned}$$

On the other hand, when $a_2 = q_2$ and $a_1 < q_1$, we need to solve the following two equations

$$\begin{aligned} 54 - (7 + 6k)q_1 - 6q_2 &= 0 \\ 45 - 5q_1 - (11 + 5k)q_2 &= 0, \end{aligned}$$

and we obtain

$$\begin{aligned}
q_1^{**} &= \frac{54\ (6+5\,k)}{47+k\ (101+30\,k)}\\
a_1^{**} &= \frac{5}{6}\hat{q}_1 = \frac{45\ (6+5\,k)}{47+k\ (101+30\,k)}\\
q_2^{**} &= a_2^{**} = \frac{45\ (1+6\,k)}{47+k\ (101+30\,k)}
\end{aligned}$$

Now we can state that the second stage equilibrium output and abatements are

$$\begin{aligned}
&(a)\quad (\tilde{q}_1, \tilde{a}_1, \tilde{q}_2, \tilde{a}_2) \text{ with } \tilde{a}_1 = \tilde{q}_1, \text{ if } t \leq t_1\\
&(b)\quad (q_1^*, a_1^*, q_2^*, a_2^*), \text{ if } t_1 < t < t_2\\
&(c)\quad (q_1^{**}, a_1^{**}, q_2^{**}, a_2^{**}) \text{ with } a_2^{**} = q_2^{**}, \text{ if } t_2 \leq t.
\end{aligned} \tag{E2.6}$$

Now, maximizing social welfare in the first stage with respect to t after taking into account the above abatements and outputs we get

$$t = \frac{9\ (35+k\ (117+4\,k\ (49+15\,k)))}{420+k\ (2713+k\ (4376+k\ (2161+330\,k)))} = t^* \tag{E2.7}$$

It is easy to check that $t_1 < t^*\ \forall k \geq 0$; but, (i) $t_2 < t^*$ if $0 \leq k < 1.24176$, and (ii) $t^* < t_2$ if $k > 1.24176$.

Case (i): If $k > 1.24176$, $t_1 < t^* < t_2$ and none of the constraints $a_1 \leq q_1$ and $a_2 \leq q_2$ binds. So, the second stage equilibrium outputs and abatements are given by (E2.5). In this case, with an increase in t the environmental damage will decline, if $t < \frac{9+18\,k}{5+k\,(16+5\,k)} = t^*$.

$$\frac{\partial E^*}{\partial t} = \frac{(5+k\ (16+5\,k))\ (-9+5\,t+k\ (-18+(16+5\,k)\ t))}{(7+k\ (19+6\,k))^2} < 0, \text{if } t < \hat{t}.$$

Substituting the optimal tax rate in (E2.5) we get $0 < t^* < Q^* - A^* = E'(.) = \frac{9\left(35+k\left(249+262\,k+60\,k^2\right)\right)}{420+k\,(2713+k\,(4376+k\,(2161+330\,k)))}$, i.e., optimal tax rate is less than marginal environmental damage.

Case (ii): Now if $0 \leq k < 1.24176$, we get $t_1 < t_2 < t^*$ and the constraint $a_2 \leq q_2$ would bind. Here the second stage outputs and abatements are given by (c) of (E2.6). Optimal tax rate must be such that the abatement constraint of firm 2 just binds, i.e. at which

$\hat{q}_2 = q_2^*$. This gives us

$$t = t_2 = \frac{9 + 54\,k}{47 + k\ (101 + 30\,k)} = t^{**}.$$

Clearly, in this case also, tax rate t^{**} is less than marginal environmental damage $(Q^{**} - A^{**})$.

2. Pollution tax and partial privatization

Example 3: Suppose $p = 10 - (q_1 + q_2)$, $C(q_i) = q_i + \frac{k}{2}q_i^2$, $g(a_i) = \frac{1}{10}a_i^2$ and $E(.) = \frac{1}{2}(q_1 + q_2 - a_1 - a_2)^2$.

Now, given the level of privatization (θ) and the tax rate (t), equilibrium outputs and abatements in Stage 2 are given by:

$$q_1: \quad 9 - t\,\theta + (1-\theta)\ (a_1 + a_2) - (2+k)\ q_1 - (2-\theta)\ q_2 = 0 \tag{E3.1}$$

$$a_1: \quad 5\,t\,\theta + 5\ (1-\theta)\ (q_1 + q_2 - a_1 - a_2) - a_1 = 0, \text{if } a_1 < q_1; \tag{E3.2a}$$

$$\text{otherwise, } a_1 = q_1 \text{ and } 5\,t\,\theta + 5\ (1-\theta)\ (q_1 + q_2 - a_1 - a_2) - a_1 \geq 0 \tag{E3.2b}$$

$$q_2: \quad 9 - t - q_1 - (2+k)\ q_2 = 0 \tag{E3.3}$$

$$a_2: \quad 5\,t - a_2 = 0, \text{if } a_2 < q_2; \tag{E3.4a}$$

$$\text{otherwise, } a_2 = q_2 \text{ and } 5\,t - a_2 \geq 0 \tag{E3.4b}$$

Therefore, the problem of the government in Stage 1 can be written as follows.

$$\underset{t,\theta}{Max}\ W(q_1(t,\theta),\, q_2(t,\theta),\, a_1(t,\theta),\, a_2(t,\theta)) \tag{E3.5}$$

subject to the constraints

$(E3.1)$, $[(E3.2a)$ or $(E3.2b)]$, $(E3.3)$ and $[(E3.4a)$ or $(E3.4b)]$

Assuming $a_1 < q_1$ and $a_2 < q_2$, from (E3.1), (E3.2a), (E3.3) and (E3.4a) we get,

$$\begin{aligned}
q_1 &= \frac{-45 - 17\,t + 18\,(2+t)\,\theta + k\,(-54 - 5\,t + 45\,\theta + 6\,t\,\theta)}{-7 - k\,(19 + 6\,k) - 6\,\theta + 5\,k\,(2+k)\,\theta + 5\,(2+k)\,\theta^2}\\
q_2 &= \frac{k\,(-9+t)\,(6 - 5\,\theta) + 9\,(-1 + 5\,(-1+\theta)\,\theta) + t\,(12 - \theta\,(6 + 5\,\theta))}{-7 - k\,(19 + 6\,k) - 6\,\theta + 5\,k\,(2+k)\,\theta + 5\,(2+k)\,\theta^2} \qquad \text{(E3.6)}\\
a_1 &= \frac{5\,(-9 + k^2\,t\,(5 - 6\,\theta) + 9\,\theta^2 + t\,(5 + (5 - 13\,\theta)\,\theta) - 2\,k\,(9 - 9\,\theta + t\,(-8 + \theta\,(7 + 3\,\theta))))}{-7 - k\,(19 + 6\,k) - 6\,\theta + 5\,k\,(2+k)\,\theta + 5\,(2+k)\,\theta^2}\\
a_2 &= 5\,t
\end{aligned}$$

Alternatively, assuming $a_1 < q_1$ but $a_2 = q_2$, from (E3.1), (E3.2a), (E3.3) and (E3.4b) we get,

$$\begin{aligned}
q_1 &= \frac{-6\,(9 + 9\,k + t) + (45 + 7\,t + k\,(45+t))\,\theta}{-8 - k\,(19 + 6\,k) - 5\,\theta + 5\,k\,(2+k)\,\theta + 5\,(2+k)\,\theta^2}\\
a_1 &= \frac{-5\,(9 + 9\,k + t + (-9\,(1+k) + k\,(3+k)\,t)\,\theta + (2+k)\,t\,\theta^2)}{-8 - k\,(19 + 6\,k) - 5\,\theta + 5\,k\,(2+k)\,\theta + 5\,(2+k)\,\theta^2} \qquad \text{(E3.7)}\\
q_2 = a_2 &= \frac{-9 + 7\,t + k\,(-9+t)\,(6 - 5\,\theta) - (45+t)\,\theta - 5\,(-9+t)\,\theta^2}{-8 - k\,(19 + 6\,k) - 5\,\theta + 5\,k\,(2+k)\,\theta + 5\,(2+k)\,\theta^2}
\end{aligned}$$

Case 1: $k = 0$

The socially optimal (symmetric) $q = 4.125$ and $a = 8.25$ and social welfare is $SW = 39.188$.

If $a_1 < q_1$ and $a_2 < q_2$, solution of the Stage 1 problem (E3.5), subject to the constraints (E3.6), is given by $t = 0.75$ and $\theta = 0$; and the corresponding outputs and abatement levels are $q_1 = 8.25$, $q_2 = 0$, $a_1 = 3.75$ and $a_2 = 3.75$. Clearly, $a_2(t = 0.75, \theta = 0) > q_2(t = 0.75, \theta = 0)$, which contradicts (E3.4a). Therefore, if $k = 0$, the constraint $a_2 \leq q_2$ will be binding.

Now, if the government chooses t and θ such that firm 2 finds it optimal to set $a_2 = q_2$, $(t,\ \theta)$ pair must satisfy $\frac{\partial \pi_2}{\partial a_2} \geq 0$, since $\frac{\partial^2 \pi_2}{\partial a_2^2} < 0$. It is easy to check that, if $k = 0$, $\frac{\partial \pi_2}{\partial a_2} = 0 \Rightarrow t = \frac{9\,(-1+5\,(-1+\theta)\,\theta)}{-47+\theta\,(-24+55\,\theta)} = T_1$, say, using the expression for a_2 from (E3.7). Also, in this case, we have $\frac{\partial}{\partial t}[\frac{\partial \pi_2}{\partial a_2}] = \frac{47+(24-55\,\theta)\,\theta}{8+5\,(1-2\,\theta)\,\theta} > 0, \forall \theta \in [0, 1]$. Therefore, we must have, $t \geq T_1$

for $\frac{\partial \pi_2}{\partial a_2} \geq 0$ to be satisfied. Now, solving the problem $Max_t\ W(.)$ subject to (E3.7), we get $t = \frac{-9\,(28+\theta\,(-39+5\,\theta\,(29+\theta\,(-61+20\,\theta))))}{84+2\,\theta\,(-12+\theta\,(153+5\,\theta\,(41+75\,\theta)))} = T_2$, say. Clearly, $T_2 < T_1$. So, we must have $t = T_1$.

Now, solving the problem $Max_\theta\ W(.)$, subject to the constraints (E3.7) and $t = T_1$, we get $\theta^* = 0.102209$, which is the optimal level of privatization in case of $k = 0$. Therefore, if $k = 0$, the equilibrium tax rate, outputs and abatements are as follows. $t^* = T_1|_{\theta=\theta^*} = 0.268611$, $q_1^* = 6.04527$, $a_1^* = 4.96893\,(< q_1^*)$, $a_2^* = q_2^* = 1.34306$. In equilibrium, profit of firm 2 is 1.98418, social welfare is 35.9726 and marginal environmental damage is 1.07634 $(> t^*)$.

Case 2: $k = 1$

Here socially optimal $q = 2.83, a = 2.57$ and $SW = 26.265$.

In this case also the constraint $a_2 \leq q_2$ is binding. To illustrate it, note that, if $a_1 < q_1$ and $a_2 < q_2$, solution of the Stage 1 problem (E3.5), subject to the constraints (E3.6), for $k = 1$ is given by $t = 0.382106$ and $\theta = 0.193162$; and the corresponding outputs and abatement levels are such that $q_1 = 3.0297 > a_1 = 2.46294$, but $q_2 = 1.86272 < a_2 = 1.91053$.

Now, if $k = 1$, $\frac{\partial \pi_2}{\partial a_2} = 0 \Rightarrow t = \frac{9\left(-7+5\,\theta^2\right)}{-178+8\,\theta\,(7+10\,\theta)} = T_3$, say, using the expression for a_2 from (E3.7). Also, $\frac{\partial}{\partial t}[\frac{\partial \pi_2}{\partial a_2}]|_{k=1} = \frac{-178+8\,\theta\,(7+10\,\theta)}{-33+5\,\theta\,(2+3\,\theta)} > 0\ \forall \theta \in [0, 1]$. It implies that, we must have, $t \geq T_3$ for $\frac{\partial \pi_2}{\partial a_2} \geq 0$ to be satisfied along with $a_2 = q_2$. Therefore, firm 2 would completely abate pollution, i.e., it would set $a_2 = q_2$, if $t \geq T_3$. Now, solving the problem $Max_t\ W(.)$ subject to (E3.7), we get $t = \frac{18\,(182+\theta\,(-259+5\,\theta\,(10+\theta\,(-43+10\,\theta))))}{1469+\theta\,(-1356+\theta\,(3746+5\,\theta\,(1012+325\,\theta)))} = T_4$, say. Upon inspection, we find that $T_4 < T_3$, if $\theta > 0.473404$; otherwise, if $0 \leq \theta < 0.473404$, $T_3 < T_4$.

Now, solving the problem $Max_\theta\ W(.)$, subject to the constraints (E3.7) and $t = T_4$, we get $\theta = 0.546929$. But, if $\theta = 0.546929$, $T_4 < T_3$. It implies that we must have $t = T_3$. Therefore, the optimal value of θ is given by the solution of the problem: $Max_\theta\ W(.)$, subject to the constraints (E3.7) and $t = T_3$. Solving this problem, we get $\theta = 0.194109 = \theta^*$. Therefore, if $k = 1$, the equilibrium level of privatization, tax rate, outputs and abatements are as follows. $\theta^* = 0.194109$, $t = T_3|_{\theta=\theta^*} = 0.373544$, $q_1^* = 3.02329 > a_1^* = 2.49426$ and $q_2^* = a_2^* = 1.86772$. In equilibrium, the profit of firm 2, social welfare and marginal environmental damage are 5.58142, 24.6329 and 0.529033, respectively. Clearly, in equilibrium, tax rate is less than the marginal environmental damage.

Case 3: $k = 2$

In this case socially optimal $q = 2.152, a = 1.956$ and $SW = 19.74$.

Here, the constraint $a_2 \leq q_2$ is not binding, and the solution of problem (E3.5), subject to the constraints (E3.6), gives the equilibrium values of t and θ as follows. $\theta^* = 0.175919$ and $t^* = 0.306122$. The equilibrium outputs, abatements, profit of firm 2 and social welfare are $q_1^* = 2.20327$, $q_2^* = 1.62265$, $a_1^* = 1.89963(< q_1^*)$, $a_2 = 1.53061(< q_2^*)$, $\pi_2^* = 5.50027$ and $W^* = 18.9536$, respectively. Here also, the equilibrium tax rate is less than marginal environmental damage (0.395682).

3. Foreign ownership and social optimality

In this section we provide an example where one firm is partially foreign-owned; however for simplicity we abstract from abatement or externality. The example can be easily extended to those cases. The main objective of this example is to show that when one firm is foreign, social optimality would require the foreign firm's output to be set very low. In particular, if the private firm is fully foreign-owned then its output should be set zero.

Suppose the demand curve is linear: $p = B - (q_1 + q_2)$, production technologies are identical as $C(q_i) = c\frac{q_i^2}{2}$. Firm 2 is foreign owned to the extent of μ.

Following Eqs. (30) and (32) we solve the socially optimal outputs as:

$$
\begin{aligned}
q_1 &= a\left[\frac{(1-\mu)(1+c) - \mu - (1-\mu)^2}{(1-\mu)(1+c)^2 - \mu(1+c) - (1-\mu)^2}\right] \\
q_2 &= \frac{a(1-\mu)}{[(1+c)(1-\mu) - \mu]}\left[\frac{(1-\mu)(1+c)^2 + \mu - (1+c)}{(1-\mu)(1+c)^2 - \mu(1+c) - (1-\mu)^2}\right].
\end{aligned}
$$

If $\mu = 1$, we have $q_2 = 0$, $q_1 = a/(1+c)$. On the other hand, if $\mu = 0$ we have $q_1 = q_2 = a/(2+c)$.

References

Antelo, M. and Loureiro., M. L. (2009). Asymmetric information, signaling and environmental taxes in oligopoly. *Ecological Economics*, 68(5):1430–1440.

Barcena-Ruiz, J.-C. and Garzon, M.-B. (2002). Environmental taxes and strategic delegation. *Spanish Economic Review*, 4(4):301–310.

Barcena-Ruiz, J. C. and Garzon, M. B. (2006). Mixed oligopoly and environmental policy. *Spanish Economic Review*, 8(2):139–160.

Barrett, S. (1994). Strategic environmental policy and international trade. *Journal of Public Economics*, 54(3):325–338.

Beladi, H. and Chao, C.-C. (2006). Does privatization improve the environment? *Economics Letters*, 93(3):343–347.

Bhattacharya, R. N. and Pal, R. (2010). Environmental standards as strategic outcomes: A simple model. *Resource and Energy Economics*, 32(3):408–420.

Canton, J., Stahn, H., and Soubeyran, A. (2008). Environmental taxation and vertical Cournot Oligopolies: How eco-industries matter. *Environmental and Resource Economics*, 40(3):369–382.

Cato, S. (2008). Privatization and environment. *Economics Bulletin*, 12(19):1–10.

Colla, P., Germain, M., and Van-Steenberghe, V. (2011). Environmental policy and speculation on markets for emission permits. *Economica*, Forthcoming.

Conrad, K. (1993). Taxes and subsidies for pollution-intensive industries as trade policy. *Journal of Environmental Economics and Management*, 25(2):121–135.

Damania, R. (2000). Financial structure and the effectiveness of pollution control in an oligopolistic industry. *Resource and Energy Economics*, 22(1):21–36.

deFraja, G. and Delbono, F. (1989). Alternative strategies of a public enterprise in oligopoly. *Oxford Economic Papers*, 41(2):302–11.

Fershtman, C. (1990). The interdependence between ownership status and market structure: The case of privatization. *Economica*, 57(227):319–28.

Fjell, K. and Heywood, J. S. (2004). Mixed oligopoly, subsidization and the order of firm's moves: the relevance of privatization. *Economics Letters*, 83(3):411–416.

Fujiwara, K. (2009). Environmental policies in a differentiated oligopoly revisited. *Resource and Energy Economics*, 31(3):239–247.

Jenkins, G. P. and Lamech, R. (1992). *Fiscal Policies to Control Pollution: International Experience*. Development Discussion Paper 439. Harvard Institute for International Development.

Jung, C., Krutilla, K., and Boyd, R. (1996). Incentives for advanced pollution abatement technology at the industry level: An evaluation of policy alternatives. *Journal of Environmental Economics and Management*, 30(1):95–111.

Kennedy, P. W. (1994). Equilibrium pollution taxes in open economies with imperfect competition. *Journal of Environmental Economics and Management*, 27(1):49–63.

Kumar, A. and Saha, B. (2008). Spatial competition in a mixed duopoly with one partially nationalized firm. *Journal of Comparative Economics*, 36(2):326–341.

Lehmann, P. (2011). Justifying a policy mix for pollution control: a review of economic literature. *Journal of Economic Surveys*, doi: 10.1111/j.1467-6419.2010.00628.x.

Lovei, M. (1995). *Financing Pollution Abatement: Theory and Practice*. Environment Department papers 28, Environmental economics series. World Bank.

Matsumura, T. (1998). Partial privatization in mixed duopoly. *Journal of Public Economics*, 70(3):473–483.

Naito, T. and Ogawa, H. (2009). Direct versus indirect environmental regulation in a partially privatized mixed duopoly. *Environmental Economics and Policy Studies*, 10(2-4):87–100.

Pal, R. (2010). Delegation and emission tax in a differentiated oligopoly. *The Manchester School*, Forthcoming.

Saha, B. (2009). Mixed ownership in a mixed duopoly with differentiated products. *Journal of Economics*, 98(1):25–43.

Silva, E. C. and Zhu, X. (2009). Emissions trading of global and local pollutants, pollution havens and free riding. *Journal of Environmental Economics and Management*, 58(2):169–182.

Ulph, A. (1996). Environmental policy and international trade when governments and producers act strategically. *Journal of Environmental Economics and Management*, 30(3):265–281.

Wang, H. and Chen, M. (1999). *How the Chinese system of charges and subsidies affects pollution control efforts by China's top industrial polluters.* Policy Research Working Paper 2198. Wold Bank.

Wang, L. F. and Wang, J. (2009). Environmental taxes in a differentiated mixed duopoly. *Economic Systems*, 33(4):389–396.

Wang, L. F. S., chin Wang, Y., and Zhao, L. (2009). Privatization and the environment in a mixed duopoly with pollution abatement. *Economics Bulletin*, 29(4):3112–3119.

White, M. (1996). Mixed oligopoly, privatization and subsidization. *Economics Letters*, 53:189–195.